New obstructions to symplectic embeddings

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Abstract

In this paper we establish new restrictions on symplectic embeddings of certain convex domains into symplectic vector spaces. These restrictions are stronger than those implied by the Ekeland-Hofer capacities. By refining an embedding technique due to Guth, we also show that they are sharp.

1 Introduction and main results

Consider $\mathbb{R}^{2n}$ equipped with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ and its standard symplectic form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. Let $B^{2n}(R)$ denote the ball of radius R in $\mathbb{R}^{2n}$. Gromov's Nonsqueezing Theorem states that there is no symplectic embedding of $B^{2n}(1)$ into $B^2(R) \times \mathbb{R}^{2(n-1)}$ for $R < 1$, [6]. The following analogous problem remains open.

Question 1. *What, if any, is the smallest value of $R > 0$ such that there exists a symplectic embedding of $B^2(1) \times \mathbb{R}^{2(n-1)}$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$?*

Prior to the current paper, the most that could be said is that if such an emebedding exists, then R must be at least $\sqrt{2}$. This bound is implied by the second Ekeland-Hofer capacity from [4].

In [7], L. Guth constructs new symplectic embeddings of polydiscs which represent a major breakthrough in our understanding of the following *bounded* version of Question 1.

Question 2. *What, if any, is the smallest value of $R > 0$ such that there exists a symplectic embedding of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$ for arbitrarily large $S > 0$?*

Among other things, Guth's work settles the existence issue for this question. [1)] In this setting, the second Ekeland-Hofer capacity again implies that R must be at least $\sqrt{2}$. The following improvement of this bound is the main result of this paper.

Theorem 1.1. *For any $0 < R < \sqrt{3}$ there are no symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

In fact we prove a slightly stronger result. For convenience, identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ using the complex coordinates $z_j = x_j + iy_j$. Let

$$E(1, S, \ldots S) = \left\{ (z_1, \ldots z_n) \in \mathbb{C}^n \;\middle|\; |z_1|^2 + \frac{|z_2|^2}{S^2} + \ldots + \frac{|z_n|^2}{S^2} \leq 1 \right\}$$

and denote by $\mathbb{C}P^2(R)$ the complex projective plane equipped with the symplectic form $R^2\omega_{FS}$ where ω_{FS} is the standard Fubini-Study symplectic form.

Theorem 1.2. *For any $0 < R < \sqrt{3}$ there are no symplectic embeddings of $E(1, S, \ldots S)$ into $\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

A brief outline of the proof. As in [6], Theorem 1.2 is proved via an existence theorem for certain holomorphic curves. Suppose that for any $S > 0$ there is a symplectic embedding $\phi(S)$ of $E(1, S, \ldots, S)$ into $\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}$. It suffices to show that for every positive integer $d < S/\sqrt{3}$, there exists a holomorphic plane of degree d in the (negative) symplectic completion of $(\mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}) \smallsetminus \phi(E(1, S, \ldots, S))$ whose negative end covers the shortest simple Reeb orbit on the boundary of $\phi(E(1, S, \ldots, S))$ a total of $3d - 1$ times. In particular, the symplectic area of such a curve is both positive and equal to $d\pi R^2 - (3d - 1)\pi$ and so the existence of these curves implies that $R^2 > (3d - 1)/d$ for all $d > 0$.

To prove this existence result we use a cobordism argument to reduce it to an equivalent problem for curves in a four dimensional symplectic manifold. Starting with well known results concerning holomorphic spheres of degree d in $\mathbb{C}P^2$, we then settle this alternative existence problem using automatic regularity theorems, the compactness theorem for splittings from [2], and several new techniques, many of which involve families of holomorphic curves

[1)] Before Guth's work it was commonly thought that the answer to Questions 1 and 2 was that no such R exists.

with varying point constraints. (In fact, we prove the existence problem in dimension four first, and then we apply it in the manner described above.)

Sharpness. We also prove that Theorem 1.1 is sharp in the following sense.

Theorem 1.3. *For any $R > \sqrt{3}$ there exist symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(R) \times \mathbb{R}^{2(n-2)}$ for all $S > 0$.*

The proof of Theorem 1.3 involves a refinement of Guth's embedding procedure from [7].

Remark 1.4. We do not know whether there exists a symplectic embedding of $B^2(1) \times B^{2(n-1)}(S)$ into $B^4(\sqrt{3}) \times \mathbb{R}^{2(n-2)}$ for all large S.

1.1 A related result

Our approach also allows us to settle the following related problem.

Question 3. *What are the smallest value of $R_1 \leq R_2$ for which there are symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^2(R_1) \times B^2(R_2) \times \mathbb{R}^{2(n-2)}$ for all $S > 0$?*

More precisely, by compactifyng $B^2(R_1) \times B^2(R_2)$ to $\mathbb{C}P^1(R_1) \times \mathbb{C}P^1(R_2)$, and replacing the study of holomorphic curves of high degree, say d, in $\mathbb{C}P^2$ by curves of degree $(d, 1)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$, an identical argument to the one used to prove Theorem 1.1 yields the following result.

Theorem 1.5. *If $R_1 < \sqrt{2}$ then there are no symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^2(R_1) \times B^2(R_2) \times \mathbb{R}^{2(n-2)}$ when S is sufficiently large.*

This restriction is again stronger than those imposed by the Ekeland-Hofer capacities which imply that no such embeddings exist, for large enough $S > 0$, when $R_1 < 1$. Moreover, Guth's embedding results again imply (this time directly) that Theorem 1.5 is sharp in the following sense.

Theorem 1.6. *([7]) For any $R > \sqrt{2}$ there exist symplectic embeddings of $B^2(1) \times B^{2(n-1)}(S)$ into $B^2(R) \times B^2(R) \times \mathbb{R}^{2(n-2)}$ for all $S > 0$.*

1.2 Organization

The next section contains some background material and the proof of the crucial existence result, Theorem 2.16, which concerns special holomorphic curves in certain four dimensional symplectic manifolds. Theorem 1.2 is then proved in Section 3. Our proof of Theorem 1.5 is completely similar and is thus omitted. In Section 4, we construct the embeddings of Theorems 1.3 and 1.6.

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2 Splitting holomorphic curves along the boundary of thin ellipsoids in $\mathbb{C}P^2$

Throughout this section we fix a real number $R > 1$ and a positive integer d.

2.1 A moduli space of curves

Let $\mathcal{J}$ be the space of smooth almost-complex structures on $\mathbb{C}P^2$ which are tame with respect to the symplectic form ω_{FS} (and hence $R^2\omega_{FS}$). For a fixed J in $\mathcal{J}$ and an ordered collection of points $p_1, \ldots, p_M$ in $\mathbb{C}P^2$, we consider the moduli space $\mathcal{M}_d(J, p_1, \ldots, p_M)$ defined as

$$\big\{(f,(y_1,\ldots,y_M)) \in C^\infty(S^2,\mathbb{C}P^2)\times(S^2)^M \mid \bar{\partial}_J f = 0,\ f(y_i) = p_i,\ [f] = dL\big\}/G,$$

where $(y_, \ldots y_M)$ is an M-tuple of pairwise distinct points in S^2, L is the class of a complex line in $\mathbb{C}P^2$ and G is the reparameterization group $\mathrm{PSL}(2,\mathbb{C})$. As is well known, see for example Section 7.4 of [16], for suitably generic J the space $\mathcal{M}_d(J, p_1, \ldots, p_M)$ is a smooth orientable manifold of dimension $2(3d - 1 - M)$. We will also make use of the following refinement of this statement.

Proposition 2.1. *For generic J in $\mathcal{J}$, the space $\mathcal{M}_d(J, p_1, \ldots, p_{3d-1})$ is a compact, zero-dimensional manifold which contains a fixed number, n_d, of points, independently of J.*

Proof. For suitably generic J, the moduli space $\mathcal{M}_d(J, p_1, \ldots, p_{3d-1})$ consists of immersed curves, that is, there are no singular curves or cusp curves. Thus the relevant Gromov-Witten invariant counting curves through the $3d-1$ points is computed by looking at immersed curves, and if these curves are counted with multiplicity then the total is independent of J. On the other hand, the automatic regularity phenomena for holomorphic spheres in dimension four (see Lemma 3.3.3 of [16]) implies that all such curves appear with the same multiplicity 1, and our conclusion follows.

□

For a generic choice of J, $\mathcal{M}_d(J, p_1, \ldots, p_{3d-2})$ is a two-dimensional manifold which can be compactified by adding finitely many cusp-curves that consist of the union of two holomorphic spheres each passing through a subset of the $3d-2$ points. In particular, if the points $p_1, \ldots, p_{3d-2}$ are in generic position, then one does not need to add cusp-curves with more than two components, nor does one need to add multiply-covered curves.

Proposition 2.2. *For $d > 2$ and a generic choice of J and the point constraints $p_1, \ldots, p_{3d-2}$, the moduli space $\mathcal{M}_d(J, p_1, \ldots, p_{3d-2})$ is connected.*

Proof. Lemma 2.3.3 of [19] yields a version of automatic regularity for holomorphic spheres of nonnegative index in a 4-manifold, which includes the possibility of singularities. In particular, the manifold $\mathcal{M}_d(J, p_1, \ldots, p_{3d-2})$ is diffeomorphic to $\mathcal{M}_d(i, p_1, \ldots, p_{3d-2})$, where i is the standard complex structure on $\mathbb{C}P^2$. Now, the space of all rational algebraic curves of degree d in $\mathbb{C}P^2$ is dominated by an open set of triples of homogeneous polynomials on $\mathbb{C}P^1$ and so is irreducible. Sets of curves through $3d-1$ points are given by intersecting this variety with an appropriate plane. It then follows from the Uniform Position Principle that any two such curves can be connected with curves passing through any subset of $3d-2$ points, see Appendix E of [8]. But any two curves of degree $d > 2$ have $d^2 > 3d-1$ points in common and so the conclusion follows.

□

2.2 A thin ellipsoid

For $a < b$, let $E(a, b)$ denote the ellipsoid

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, \frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} \leq 1 \right\}.$$

If $\frac{a^2}{b^2} \in \mathbb{R} \smallsetminus \mathbb{Q}$, then the standard Liouville form on $\mathbb{C}^2$ restricts to $\partial E(a,b)$ as a contact form $\alpha_{E(a,b)}$ which has exactly two simple Reeb orbits, γ_1 and γ_2, which lie in the planes $\{z_2 = 0\}$ and $\{z_1 = 0\}$, respectively. The action of γ_1 is πa^2 and the action of γ_2 is πb^2. The Conley-Zehnder indices of these orbits, with respect to the natural trivialization of the z_1 and z_2 coordinate planes, are given by $\mu(\gamma_1) = 3$ and $\mu(\gamma_2) = 2 + 2\left\lfloor \frac{b^2}{a^2} \right\rfloor + 1$. More generally, if $\gamma_j^{(r)}$ is the r-fold cover of γ_j, then we have

$$\mu(\gamma_1^{(r)}) = 2r + 2\left\lfloor \frac{ra^2}{b^2} \right\rfloor + 1 \tag{1}$$

and

$$\mu(\gamma_2^{(r)}) = 2r + 2\left\lfloor \frac{rb^2}{a^2} \right\rfloor + 1. \tag{2}$$

These trivialization along closed Reeb orbits will be used throughout, see Remark 2.5 below.

It follows from Theorem 2 of [18] that for $S > \sqrt{2}$ the ellipsoid $E(1/S, 1)$ can be symplectically embedded into the interior of $B^4(1)$ and hence $\mathbb{C}P^2(R)$. For a fixed degree d we now fix such an embedding of $E(1/S, 1)$ for an S satisfying

$$S > \sqrt{3}d. \tag{3}$$

In what follows, we will denote $E(1/S, 1)$ by E and will identify E with its image in $\mathbb{C}P^2(R)$.

2.3 Splitting curves in $\mathbb{C}P^2$ along ∂E

In this section we recall from [2] the structure of the limits of holomorphic spheres in $\mathbb{C}P^2$ as one splits this manifold (*stretches the neck*) along the boundary of the embedded ellipsoid E.

Let X be the vector field defined near ∂E as the symplectic dual of the Liouville form. An almost-complex structure J on $\mathbb{C}P^2$ is said to be compatible with E if the contact structure $\{\alpha_E = 0\}$ on ∂E is equal to $T(\partial E) \bigcap JT(\partial E)$, and JX is equal to the Reeb vector field of α_E. Denote by $\mathcal{J}_E$ the set of $J \in \mathcal{J}$ which are compatible with E.

For every natural number $N \in \mathbb{N}$, the union of the three pieces

$$\left(E, e^{-N}\omega_0\right),\ (\partial E \times [-N, N], d(e^{\tau}\alpha_E)),\ \text{ and }\ \left(\mathbb{C}P^2 \smallsetminus E, e^{N}\omega_{FS}\right),$$

attached along their appropriate boundary components, is a symplectic manifold $((\mathbb{C}P^2)^N, \omega^N_{FS})$. For a J in $\mathcal{J}_E$, let J^N be the continuous almost-complex structure on $(\mathbb{C}P^2)^N$ which equals J on the disjoint union of $(\mathbb{C}P^2 \smallsetminus E)$ and E, and is translation invariant on $\partial E \times [-N, N]$. To make each of the J^N smooth one must perturb J near ∂E. As noted in §3.4 of [2], the choice of this perturbation is irrelevant for the compactness theorem below. Accordingly, we will henceforth assume that this choice has been made, and that the almost-complex structures J^N are smooth.

Fix points $p_1, \dots, p_M$ in $E \subset \mathbb{C}P^2$ and consider a sequence C_N in the space $\mathcal{M}_d(J^N, p_1, \dots, p_M)$. Fix a representative curve f_N for each C_N. The following compactness theorem is proved by Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder in [2].

Theorem 2.3. *(Theorem 10.6 of [2]) There exists a subsequence of the f_N which converges to a holomorphic building,* $\mathbf{F}$.

We now describe the relevant aspects of this theorem starting with the structure of the limiting building $\mathbf{F}$. The reader is referred to [2] for the precise definitions, statements and proofs of the general compactness theorem for holomorphic curves under splittings.

2.3.1 The limits

We begin by describing a holomorphic building $\mathbf{F}$ which arises as the limit of a subsequence of the curves f_N. The domain of $\mathbf{F}$ is a nodal Riemann sphere $(\mathcal{S}, j)$ with punctures. The building $\mathbf{F}$ then consists of a collection of finite energy holomorphic maps from the collection of punctured spheres of $\mathcal{S} \smallsetminus \{\text{nodes}\}$ to one of the following three symplectic manifolds:

- $\left(E^\infty_+, \omega^\infty_+\right) = (E, \omega_0) \cup (\partial E \times [0, \infty), d(e^\tau \alpha_E))$,
- $(SE, \omega_{SE}) = (\partial E \times \mathbb{R}, d(e^\tau \alpha_E))$,
- $\left((\mathbb{C}P^2 \smallsetminus E)^\infty_-, \omega^\infty_-\right) = (\mathbb{C}P^2 \smallsetminus E, \omega_{FS}) \cup (\partial E \times (-\infty, 0], d(e^\tau \alpha_E))$.

More precisely, these target manifolds are equipped with the almost-complex structures which are induced by J and are translation invariant on the subsets which are cylinders over ∂E. The curves of $\mathbf{F}$ are then holomorphic with respect to these almost-complex structures and the complex structures on the punctured spheres induced by the structure j on $\mathcal{S}$.

Since each curve of $\mathbf{F}$ has finite energy, they are all asymptotically cylindrical near each puncture, to some multiple of either γ_1 or γ_2. If F is a curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$, then the punctures in its domain are all negative, i.e as $z \in S^2$ approaches a puncture on the domain of F, $F(z)$ takes values in $\partial E \times (-\infty, 0]$ and its $(-\infty, 0]$-component converges to $-\infty$. Similarly, each curve of $\mathbf{F}$ with image in E^{∞}_{+} has only positive punctures, and curves of $\mathbf{F}$ with image in SE have both negative and positive punctures (but not only negative punctures).

The limiting building $\mathbf{F}$ is also equipped with a *level structure*. For a building $\mathbf{F}$ of level k, this structure is encoded by a labeling of the punctured Riemann spheres of $\mathcal{S} \smallsetminus \{\text{nodes}\}$ by integers from 0 to $k+1$, called levels, such that the levels of two components which share a node, differ at most by 1. Let $\mathcal{S}_r$ denote the union of components of level r and denote by v_r the holomorphic curve of $\mathbf{F}$ with (possibly disconnected) domain $\mathcal{S}_r$. Then $v_0 : \mathcal{S}_0 \to E^{\infty}_{+}$, $v_r : \mathcal{S}_r \to SE$, for $1 \leq r \leq k$, and $v_{k+1} : \mathcal{S}_{k+1} \to (\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$. Moreover, each node shared by $\mathcal{S}_r$ and $\mathcal{S}_{r+1}$ is a positive puncture for v_r and a negative puncture for v_{r+1}, each asymptotic to the same Reeb orbit. As well, v_r extends continuously across each node within $\mathcal{S}_r$.

Lastly, we recall that the curves of $\mathbf{F}$ have two collective properties. The first of these is the fact that the sum, over components, of their virtual indices (see below) is equal to $2(3d - 1 - M)$, the deformation index of the curves f_N. To state the second collective property, we must first recall the definition of a *compactifcation* of a curve of $\mathbf{F}$. This definition depends on the target of the curve. For a curve G of $\mathbf{F}$ with image in $SE = \partial E \times \mathbb{R}$ one can write $G = (g, a)$ where g maps the domain of G to ∂E. The map g then extends to a continuous map $\overline{G}$, the compactification of G, which takes the oriented blow-up of the domain of G to ∂E such that the circle corresponding to each puncture is mapped to the closed Reeb orbit on ∂E which determines the asymptotic behavior of G near that puncture, (see Proposition 5.10 of [2]). Let F be a curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$. As described in Section 3.2 of [2], one can identify $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$ with $\mathbb{C}P^2 \smallsetminus E$ via a diffeomorphism Ψ_{-} which is the identity map away from an arbitrarily small tubular neighborhood of $\partial(\mathbb{C}P^2 \smallsetminus E)$ in $\overline{\mathbb{C}P^2 \smallsetminus E}$. One can then extend the map $\Psi_{-} \circ F$ to a smooth map $\overline{F}$ which takes the oriented blow-up of the domain of F to $\overline{\mathbb{C}P^2 \smallsetminus E}$ such that each boundary circle goes to the appropriate closed Reeb orbit. A choice of this extension $\overline{F}$ is a compactification of F. Similarly, for a curve H of $\mathbf{F}$ with image in E^{∞}_{+}, one can use a diffeomorphism $\Psi_{+} \colon E^{\infty}_{+} \to E \smallsetminus \partial E$ to define a compactification $\overline{H}$

of H as a smooth extension of $\Psi_+ \circ H$ which takes the oriented blow-up of the domain of H to E, and again takes each boundary circle to the corresponding closed Reeb orbit on ∂E. If one fixes a compactification for each of the curves in $\mathbf{F}$, then these maps must fit together to form a continuous map $\overline{\mathbf{F}}\colon S^2 \to \mathbb{C}P^2$.

2.3.2 The convergence

Let $\mathbf{F}$ be a holomorphic building of level k, as above, whose domain is the Riemann surface with nodes $(\mathcal{S}, j)$. If $\mathbf{F}$ is the limit of the holomorphic spheres f_N in the sense of [2], then there exist maps $\sigma_N : S^2 \to \mathcal{S}$ and sequences $s_N^r \in \mathbb{R}$, $r = 1, \ldots, k$, such that:

(i). The σ_N are diffeomorphisms except that they may collapse a finite collection of circles in S^2 to nodes in $\mathcal{S}$. Moreover, $\sigma_{N*} i$ converges to j away from the nodes of $\mathcal{S}$, where i is the standard complex structure on S^2.

(ii). The sequences of maps $f_N \circ \sigma_N^{-1} : \mathcal{S}_0 \to E_+^\infty$ and $f_N \circ \sigma_N^{-1} : \mathcal{S}_{k+1} \to (\mathbb{C}P^2 \smallsetminus E)_-^\infty$ converge in the C^∞_{loc}-topology to the maps v_0 and v_{k+1}, respectively. For $1 \leq r \leq k$ the maps $\psi^{s_N^r} \circ f_N \circ \sigma_N^{-1} : \mathcal{S}_r \to SE$ converge to v_r in the C^∞_{loc}-topology where $\psi^{s_N^r}$ is the diffeomorphism of $SE = \partial E \times \mathbb{R}$ which translates the $\mathbb{R}$-component by s_N^r.

Here, as is necessary, we are identifying $\partial E \times (-N, N) \subset (\mathbb{C}P^2)^N$ with an increasing sequence of domains in SE, $E \cup \partial E \times (-N, N) \subset (\mathbb{C}P^2)^N$ with an increasing sequence of domains in E_+^∞, and $\mathbb{C}P^2 \smallsetminus E \cup \partial E \times (-N, N] \subset (\mathbb{C}P^2)^N$ with an increasing sequence of domains in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$.

2.4 Finer restrictions on limiting holomorphic buildings

When the number of point constraints M is equal to $3d - 1$, one can use the special nature of the Reeb flow of α_E on ∂E, together with the regularity results of Dragnev from [3], to obtain more restrictions on the curves comprising our limiting holomorphic buildings. Throughout this section we will consider curves of a limiting building which have connected domains.

Proposition 2.4. *Let* $\mathbf{F}$ *be a holomorphic building which is the limit of a sequence of curves* f_N *representing classes in* $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$*. For a generic choice of* J *in* $\mathcal{J}_E$ *and the constraint points* $p_1, \ldots, p_{3d-1}$ *we have:*

(i). The curves in $\mathbf{F}$ *whose image lies in* $(\mathbb{C}P^2 \setminus E)^{\infty}_{-}$ *are somewhere injective, regular, and have deformation index equal to zero. Their (negative) ends are all asymptotic to multiples of* γ_1*.*

(ii). The curves in $\mathbf{F}$ *whose image lies in* SE *are all multiple covers of holomorphic cylinders over* γ_1*, and they have virtual index zero.*

(iii). The curves in $\mathbf{F}$ *whose image lies in* E^{∞}_{+} *are somewhere injective, regular, and have deformation index equal to zero (with the point constraints). Their (positive) ends are all asymptotic to multiples of* γ_1*.*

Proof. By [3], we may choose J in $\mathcal{J}_E$ so that the corresponding almost-complex structures on $(\mathbb{C}P^2 \setminus E)^{\infty}_{-}$, SE, and E^{∞}_{+} are regular in the sense that for all somewhere injective curves which have finite energy and genus zero, the deformation index of the curve is equal to its virtual index. We choose the points $p_1, \ldots, p_{3d-1}$ so that they are in general position, as constraints, for the spaces $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$ as well as for the moduli spaces of somewhere injective finite energy pseudo-holomorphic curves of genus zero in E^{∞}_{+}.

Consider a curve F of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \setminus E)^{\infty}_{-}$. Suppose that F has s_1^- negative ends asymptotic to multiples of γ_1, and s_2^- negative ends asymptotic to multiples of γ_2. If the i^{th} negative end covering γ_1 does so a_i^- times, and the i^{th} negative end covering γ_2 does so b_i^- times, then the virtual deformation index of F is

$$\mathrm{index}(F) = (-1)(2 - s_1^- - s_2^-) + 2c_1(F) - \sum_{i=1}^{s_1^-} \mu(\gamma_1^{(a_i^-)}) - \sum_{i=1}^{s_2^-} \mu(\gamma_2^{(b_i^-)}).$$

Remark 2.5. The index formulas used in this paper all involve Chern numbers of the finite energy curves. Finite energy curves do not give closed cycles in the ambient manifold, say X, however they are asymptotic at their ends to periodic orbits of Reeb flows and so can be compactified to give 2-dimensional cycles with boundary, which can always be perturbed to be defined by immersions. The definition of the Chern number depends upon a choice of trivialization of TX along this boundary, and that trivialization must of course be compatible with the one used in Section 2.2 to define our Conley-Zehnder indices.

To be precise, in all cases considered in this paper our Reeb orbits γ will lie in the boundary of an ellipsoid E in $\mathbb{C}^n$. Thus we have a standard symplectic trivialization of $T(X)|_\gamma$ coming from restriction of the standard trivialization on $\mathbb{C}^n$. The action of the derivative of the Reeb flow (extended to act trivially on the normal vector to $\partial E|_\gamma$) induces a family of symplectomorphisms $\eta_t \in \mathrm{S}ymp(\mathbb{C}^n)$ for $0 \le t \le L$, where L is the length of the Reeb orbit. We then define $\mu(\gamma)$ following [17], as was done above in Section 2.2. Suppose that X has real dimension $2n$ and is equipped with an almost-complex structure J. The determinant line bundle $\Lambda^n(X, J)$ has a standard section S over E, again using our trivialization. In the above formula $c_1(F)$ can be defined to be the number of zeros (counted with multiplicity) of a section of $\Lambda^n(X, J)|_F$ which agrees with S over γ.

When applied to finite energy curves lying in E^∞_+ or SE these choices imply automatically that the Chern numbers vanish. For curves lying in $(X \setminus E)^\infty_-$ the Chern number is the same as the usual Chern number, $\langle c_1(X, J), [\widehat{F}]\rangle$, where $[\widehat{F}]$ is the homology class represented by the cycle $\widehat{F}$ formed by gluing the appropriate discs in E to the compactification of F. In the case when $X = \mathbb{C}P^2$, by Poincaré duality this is just 3 times the intersection number of F with the line at infinity in the complex projective space.

By the iteration formulas (1) and (2) , the index formula above simplifies to

$$\mathrm{index}(F) = -2 + 2c_1(F) - 2\sum_{i=1}^{s_1^-}(a_i^- + \lfloor a_i^-/S^2\rfloor) - 2\sum_{i=1}^{s_2^-}(b_i^- + \lfloor b_i^- S^2\rfloor). \quad (4)$$

As described above, the Chern number $c_1(F)$ is equal to three times the intersection number of F with the line at infinity. But by positivity of intersection this intersection number is bounded above for each component by the intersection number of degree d curves with the line at infinity. Hence, $c_1(F)$ is at most $3d$, the Chern class of degree d curves in $\mathbb{C}P^2$. As well, the bound on S from (3) implies that $\lfloor b_i^- S^2\rfloor \ge b_i^- 3d$, and so

$$\mathrm{index}(F) \le -2 + 6d - 2\sum_{i=1}^{s_1^-} a_i^- - (2+6d)\sum_{i=1}^{s_2^-} b_i^-.$$

Together with (4), this expression leads immediately to the following result.

Lemma 2.6. *If a holomorphic curve F of* $\mathbf{F}$ *with image in* $(\mathbb{C}P^2 \setminus E)^\infty_-$ *has nonnegative virtual index, then* $c_1(F) > 0$, $s_2^- = 0$ *and* $\sum_{i=1}^{s_1^-} a_i^- \leq 3d - 1$. *Moreover,*

$$\text{index}(F) = -2 + 2c_1(F) - 2\sum_{i=1}^{s_1^-} a_i^-.$$

The Fredholm theory gives regularity for somewhere injective curves with respect to generic almost-complex structures. We will need the following proposition to describe possible finite energy curves which are not somewhere injective.

Proposition 2.7. *Let X be a symplectic manifold perhaps having cylindrical ends and equipped with a compatible almost-complex structure as described above. A finite energy holomorphic curve $u : \Sigma \to X$ is either somewhere injective or there exists a proper holomorphic map $\phi : \Sigma \to \Sigma'$ and a somewhere injective curve $u' : \Sigma' \to X$ such that $u = u' \circ \phi$.*

Proof. This follows almost exactly as in the case of closed holomorphic curves, see for example [16], Proposition 2.5.1, at least if we assume that the Reeb flow is nondegenerate. Then our finite energy curves converge asymptotically to cylinders over closed Reeb orbits and in particular have only finitely many critical points, see [11]. The proof has already been adapted to finite energy planes in [12], Theorem 6.2, in which case the map ϕ is shown to be polynomial. At least in the nondegenerate case this proof applies equally well to finite energy curves with multiple ends and perhaps higher genus (although of course in the higher genus case ϕ may not necessarily be polynomial). □

In the situation at hand we may therefore suppose that F is an r-fold cover of a simple curve $\widetilde{F}$. It follows from the regularity of J, that $\text{index}(\widetilde{F}) \geq 0$. Lemma 2.6 then implies that $c_1(\widetilde{F}) > 0$, $\tilde{s}_2^- = 0$, $\sum_{i=1}^{\tilde{s}_1^-} a_i^- \leq 3d - 1$, and

$$\text{index}(\widetilde{F}) = -2 + 2c_1(\widetilde{F}) - 2\sum_{i=1}^{\tilde{s}_1^-} \tilde{a}_i^-.$$

Hence, for F we have $s_2^- = 0$ and

$$\text{index}(F) \geq -2 + r(\text{index}(\widetilde{F}) + 2) \geq 0. \tag{5}$$

Thus, the hypothesis of Lemma 2.6 also holds for each curve F of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \setminus E)^\infty_-$ and we have

Lemma 2.8. *Let F be a curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$. Then $c_1(F) > 0$ and the ends of F are all asymptotic to some multiple of γ_1 and cover γ_1 at most $3d-1$ times. The virtual index of F is nonnegative and is strictly positive when F is a multiple cover.*

Now consider a curve G of $\mathbf{F}$ whose image lies in the symplectization SE. Since none of the curves of $\mathbf{F}$ in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ have negative ends asymptotic to multiples of γ_2, it follows from the existence of the map $\overline{\mathbf{F}}$ that, at least if G is of the top level k, the positive ends of G can only be asymptotic to multiples of γ_1. Suppose that G has s_1^+ such ends, and that the i^{th} one covers γ_1 a total of a_i^+ times. As established above, each curve of $\mathbf{F}$ in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ has at most $3d-1$ negative ends when counted with multiplicity, and there are at most d such curves since each one has a positive Chern number. Hence

$$\sum_{i=1}^{s_1^+} a_i^+ \leq d(3d-1) < S^2. \tag{6}$$

Suppose that G has s_1^- negative ends asymptotic to multiples of γ_1 with the i^{th} such end covering this orbit b_i^- times, and G has s_2^- negative ends asymptotic to multiples of γ_2 with the i^{th} such end covering γ_2 a total of c_i^- times. Then, by Stokes' Theorem we have

$$\begin{aligned} 0 &\leq \int_G d\alpha_E \\ &= \frac{\pi}{S^2}\left(\sum_{i=1}^{s_1^+} a_i^+ - \sum_{i=1}^{s_1^-} b_i^-\right) - \pi\left(\sum_{i=1}^{s_2^-} c_i^-\right) \\ &\leq \frac{\pi}{S^2}\sum_{i=1}^{s_1^+} a_i^+ - \pi\sum_{i=1}^{s_2^-} c_i^- \\ &\leq \frac{\pi d(3d-1)}{S^2} - \pi\sum_{i=1}^{s_2^-} c_i^-. \end{aligned}$$

Our choice of S satisfying (3) implies that

$$\sum_{i=1}^{s_2^-} c_i^- \leq \frac{d(3d-1)}{S^2} < 1,$$

and so $s_2^- = 0$. Integrating $d\alpha_E$ over G once again, we now have

$$\sum_{i=1}^{s_1^+} a_i^+ - \sum_{i=1}^{s_1^-} b_i^- \geq 0. \tag{7}$$

Hence the total number of positive ends of G is no less than the total number of its negative ends and by (6) and (7) we have

$$\begin{aligned}
\text{index}(G) &= -2 + s_1^+ - s_1^- + \sum_{i=1}^{s_1^+} \mu(\gamma_1^{(a_i^+)}) - \sum_{i=1}^{s_1^-} \mu(\gamma_1^{(b_i^-)}) \\
&= -2 + 2s_1^+ + 2\sum_{i=1}^{s_1^+} (a_i^+ + \lfloor a_i^+/S^2 \rfloor) - 2\sum_{i=1}^{s_1^-} (b_i^- + \lfloor b_i^-/S^2 \rfloor) \\
&= 2(s_1^+ - 1) + 2\left(\sum_{i=1}^{s_1^+} a_i^+ - \sum_{i=1}^{s_1^-} b_i^- \right) \\
&\geq 0.
\end{aligned}$$

Hence, the virtual index of G is strictly positive unless $s_1^+ = 1$ and $\sum_{i=1}^{s_1^+} a_i^+ = \sum_{i=1}^{s_1^-} b_i^-$. This condition is equivalent to the curve being a multiple cover of a cylinder over γ_1. As G has no negative ends asymptotic to γ_2 then same conclusions apply by induction to lower level curves mapping to SE. To summarize, we have

Lemma 2.9. *Let G be a curve of $\mathbf{F}$ with image in the symplectization SE. The positive and negative ends of G are all asymptotic to some multiple of γ_1 and the positive ends cover γ_1 at least as many times as the negative ends. The virtual index of G is nonnegative and is strictly positive unless G has one positive end and is a multiple cover of a cylinder over γ_1.*

Finally, we consider a curve H of $\mathbf{F}$ whose image is in E_+^∞. Our analysis of the curves G above implies that none of the positive ends of H are asymptotic to multiples of γ_2. Suppose that H has s_1^+ positive ends asymptotic to

multiples of γ_1 with the i^{th} such end covering this orbit b_i^+ times. Then

$$\begin{aligned}\text{index}(H) &= (-1)(2-s_1^+) + \sum_{i=1}^{s_1^+} \mu(\gamma_1^{(b_i^+)}) + -2M \\ &= -2 + 2s_1^+ - 2M + 2\sum_{i=1}^{s_1^+}(b_i^+ + \lfloor b_i^+/S^2 \rfloor)\end{aligned}$$

where $M \leq 3d-1$ is the number of point constraints achieved by the curve H. The fact that the negative ends of the collection of curves of $\mathbf{F}$ in $(\mathbb{C}P^2 \setminus E)_-^\infty$ cover γ_1 at most $d(3d-1)$ times, together with the fact that positive ends of the curves in SE cover γ_1 at least as many times as their negative ends, implies that

$$\sum_{i=1}^{s_1^+} b_i^+ \leq d(3d-1). \tag{8}$$

Hence

$$\text{index}(H) = -2 + 2s_1^+ - 2M + 2\sum_{i=1}^{s_1^+} b_i^+. \tag{9}$$

Arguing as above, the curve H can be realized as an r-fold cover of a somewhere injective curve $\widetilde{H}$ with the same point constraints. By regularity

$$\text{index}(\widetilde{H}) = -2 + 2\tilde{s}_1^+ - 2M + 2\sum_{i=1}^{\tilde{s}_1^+} \tilde{b}_i^+ \geq 0. \tag{10}$$

Since $\sum_{i=1}^{s_1^+} b_i^+ = r\sum_{i=1}^{\tilde{s}_1^+} \tilde{b}_i^+$ and $s_1^+ \geq \tilde{s}_1^+$, we then have

$$\text{index}(H) \geq \text{index}(\widetilde{H}) + (r-1)2\sum_{i=1}^{\tilde{s}_1^+} \tilde{b}_i^+ \geq 0. \tag{11}$$

Lemma 2.10. *Let H be a curve of $\mathbf{F}$ with image in E_+^∞. The positive ends of H are asymptotic to some multiple of γ_1. The virtual index of H is nonnegative and is strictly positive if H is multiply covered.*

Note that the deformation index of the curves in $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$ is zero, and so the sum of the virtual indices of the curves of $\mathbf{F}$ is also zero.

It follows from Lemmas 2.8, 2.9, and 2.10, that every curve of $\mathbf{F}$ must have virtual index zero. The same three lemmas then yield the three statements of Proposition 2.4.

□

Remark 2.11. The third statement of Proposition 2.4 implies that the curves in $\mathbf{F}$ with image in E_+^∞ must each pass through at least one of the point constraints. In particular, these curves are regular with deformation index equal to zero, and unconstrained regular curves must have deformation index at least two. More generally, curves in E_+^∞ asymptotic to γ_1^r have an unconstrained index $2r$ and so must pass through at least r of the point constraints.

For any integer $l > 0$, one can symplectically embed l disjoint balls of radius one into the ellipsoid $E(1, \sqrt{l})$, [20]. Hence, for S satisfying (3), we may assume that the points $p_1, \ldots, p_{3d-1}$ lie at the center of disjoint balls in $E = E(1/S, 1)$ which have radii slightly less than $1/S$ and whose closures lie in the interior of E. Let $J \in \mathcal{J}_E$ be an almost-complex structure which agrees with the standard complex structure in the balls around the points p_i, and induces almost-complex structures on $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$, SE, and E_+^∞ which are regular for somewhere injective, finite energy curves of genus zero. Consider again a sequence of curves f_N which represent classes in $\mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$ and converge, in the sense of [2], to a holomorphic building $\mathbf{F}$. For this new choice of J we get the following refinements of Proposition 2.4.

Lemma 2.12. *With J as above, there is exactly one curve of $\mathbf{F}$ whose image lies in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$. It has exactly $3d-1$ negative ends when counted with multiplicity. Moreover, the curves of $\mathbf{F}$ with image in SE each have exactly one positive puncture and the curves of $\mathbf{F}$ with image in E_+^∞ are all holomorphic planes.*

Proof. We first show that the negative ends of the curves of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$ cover γ_1 at least $3d-1$ times. Let G_E be the collection of curves of $\mathbf{F}$ with images in either SE or E_+^∞. Denote by $\overline{G}_E$ the map to E formed by fitting together the compactifications of the curves of G_E. By the existence of the map $\overline{\mathbf{F}} \colon S^2 \to \mathbb{C}P^2$, the ends of $\overline{G}_E$ cover γ_1 the same number of times, say r, as the curves of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$. Now, the symplectic area of $\overline{G}_E$ is $r\pi/S^2$. On the other hand, since the compactifications of curves E_+^∞

are holomorphic away from ∂E, the monotonicity theorem and our choice of J imply that the intersections of $\overline{G}_E$ with the balls centered at the points p_j have symplectic area arbitrarily close to $\frac{\pi(3d-1)}{S^2}$. Hence, $r \geq 3d-1$.

So, the curves of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$ have a total Chern number of $3d$ and, collectively, their negative ends cover γ_1 at least $3d-1$ times. As shown in the proof of Proposition 2.4, the deformation index of a curve F in $\mathbf{F}$ whose image lies in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$ is given by

$$\text{index}(F) = -2 + 2c_1(F) - 2\sum_{i=1}^{s_1^-} a_i^-.$$

where s_1^- represents the number of negative ends and a_i^- is the number of times the i^{th} negative end covers γ_1. If there are K such curves, then their total deformation index is at most

$$-2K + 2(3d) - 2(3d-1) = -2K + 2.$$

Since the total index must be nonnegative we have $K = 1$. Hence, there is exactly one curve, say F, of $\mathbf{F}$ in $(\mathbb{C}P^2 \smallsetminus E)^{\infty}_{-}$, and F has index zero and exactly $3d-1$ negative ends, when counted with multiplicity.

The remaining statements of Lemma 2.12 follows easily from the first as F is a limit of genus 0 curves.

□

2.5 Holomorphic curves with varying point constaints

In this section we prove two results involving families of (moduli spaces of) curves with varying point constraints. Let $p_i(t)$ be a set of paths in E for $1 \leq i \leq 3d-1$ and $0 \leq t \leq 1$. For each $N \in \mathbb{N}$ set

$$\mathcal{N}_N = \{C | C \in \mathcal{M}_d(J^N, p_1(t), \ldots, p_{3d-1}(t)) \text{ for some } t \in [0,1]\},$$

where J^N is the almost-complex structure on $((\mathbb{C}P^2)^N, \omega_{FS}^N)$ defined by $J \in \mathcal{J}_E$ in the manner described in Section 2.3.

Proposition 2.13. *For a given fixed N and almost-complex structure $J \in \mathcal{J}_E$, there exists an open, dense set of paths $p_i(t)$ for which the moduli spaces $\mathcal{N}_N$ are compact 1-dimensional manifolds and the natural projection onto $[0,1]$ is a covering map of degree n_d.*

Proof. The set of nonimmersed curves is closed, and nonempty only with respect to a codimension 2 collection of point constraints. Therefore for an open dense set of paths we may assume that all curves in $\mathcal{N}_N$ are immersed. Automatic regularity for holomorphic spheres in dimension four, see Lemma 3.3.3 of [16], implies that immersed spheres with nonnegative virtual deformation index are in fact regular. The same is true for moduli spaces of curves satisfying point constraints, as can be seen by identifying curves passing through the points with unconstrained curves in a suitable blow-up at the points. Alternatively, and more directly, we can apply Theorem 4.5.3 of [19] which gives regularity for all curves in a generic path.

□

The following result will be used several times in the proof of Theorem 1.1.

Proposition 2.14. *For a suitable choice of almost-complex structure $J \in \mathcal{J}_E$, and for a generic family of paths $p_i(t)$, the conclusions of Proposition 2.13 hold for all N. Further, let $f_N(t)$, for $t \in [0,1]$, be a generic 1-parameter family of curves representing a component of $\mathcal{N}_N$. If the subsequences $f_{N_j}(0)$ and $f_{N_j}(1)$ both converge, then the top level curves of the two corresponding limiting holomorphic buildings, that is the curves which map into $(\mathbb{C}P^2 \setminus E)^\infty_-$, have identical images.*

Proof. We begin by describing our choice of generic data. We choose $J \in \mathcal{J}_E$ so that the corresponding limiting almost-complex structures on $(\mathbb{C}P^2 \setminus E)^\infty_-$ and E^∞_+ are regular for somewhere injective finite energy pseudo-holomorphic curves of genus zero. We then choose the paths $p_i(t)$ so that the moduli spaces $\mathcal{N}_N$ are all regular. It follows from Proposition 2.13, that this corresponds to choosing the paths $p_i(t)$ from a countable, over N, intersection of open dense subsets of families of paths. We then restrict the choice of paths again so that for all $t \in [0,1]$ the points $p_i(t)$ are in general position, as constraints, for the moduli spaces of somewhere injective finite energy pseudo-holomorphic curves of genus zero in E^∞_+. We note that the relevant moduli spaces of curves in E^∞_+ have even unconstrained indices. Therefore the general position condition can be stated equivalently as follows, we require that for any $t \in [0,1]$ and any finite energy curve in E^∞_+ of genus zero and deformation index $2k$, the curve passes through at most k of the $\{p_i(t)\}$.

We now verify that this last condition is also generic, that is, is satisfied by families of $3d-1$ paths in an open, dense subset of the space of all such families of paths. Since finite intersections of dense open sets are dense and

open, it suffices to show that there is an open dense set of paths $\{p_i(t)\}_{i=1}^{k+1}$ such that no genus 0 finite energy curve in E_+^∞ of index $2k$ passes through all $p_i(t)$ for any t. Consider a moduli space $\mathcal{B}$ of such curves. There is a smooth evaluation map $\mathcal{B}^{k+1} \times [0,1] \to E^{k+1} \times [0,1]$ acting trivially on the second factor, where $\mathcal{B}^{k+1}$ denotes the corresponding moduli space of curves equipped with $k+1$ marked points and E^{k+1} is $k+1$ copies of E. Now, $\mathcal{B}^{k+1}$ has dimension $2k+2(k+1)$ while E^{k+1} has dimension $4(k+1)$. Thus by Sard's theorem the image misses a generic submanifold of codimension $2k+2(k+1)+1+1$, that is, dimension 1. Thus an open, dense set of paths $(p_1(t), \dots, p_{k+1}(t), t)$ will avoid the image of the evaluation map, which is the same as saying that no curve in the moduli space will pass through all of the points, for any t.

With these choices in place, the conclusions of Proposition 2.4 hold for any holomorphic building which arises as the limit of a subsequence of curves the form $f_N(t_N)$. Assume now that the top level curves of the limiting buildings for $f_{N_j}(0)$ and $f_{N_j}(1)$ have distinct images. Below, we derive from this assumption the existence of a subsequence of curves of the form $f_N(t_N)$ that converges to a holomorphic building with a top level curve of virtual index greater than zero. This will contradict the first assertion of Proposition 2.4.

It follows from our choice of J that every finite energy J-holomorphic curve in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$ with genus zero and virtual index zero is regular. (For this, we recall from equation 5 that for regular J any curve which is a nontrivial multiple cover necessarily has deformation index at least 2.) Hence there are only finitely many images of such curves of degree bounded by d, say $I_1, \dots, I_l$. Set

$$\overline{I}_j = I_j \cap \left(\overline{\mathbb{C}P^2 \smallsetminus E}\right).$$

By the unique continuation principle, $\overline{I}_1, \dots, \overline{I}_l$ are distinct compact subsets of $\overline{\mathbb{C}P^2 \smallsetminus E}$.

Let $\mathbf{F}_0$ and $\mathbf{F}_1$ be the limiting buildings of $f_{N_j}(0)$ and $f_{N_j}(1)$, respectively. The first statement of Proposition 2.4 implies that the top levels curves of $\mathbf{F}_0$ and $\mathbf{F}_1$ are all somewhere injective finite energy curves of genus zero with image in $(\mathbb{C}P^2 \smallsetminus E)_-^\infty$. By our assumption, the intersections of the images of these top level curves with $\overline{\mathbb{C}P^2 \smallsetminus E}$ are distinct subsets, say $\mathcal{I}_0$ and $\mathcal{I}_1$, of $\{\overline{I}_1, \dots, \overline{I}_l\}$. Fixing a metric on $\overline{\mathbb{C}P^2 \smallsetminus E}$ we consider the corresponding Hausdorff metric on compact subsets of $\overline{\mathbb{C}P^2 \smallsetminus E}$. For large j, the sets

$f_{N_j}(t)(S^2) \cap \left(\overline{\mathbb{C}P^2 \smallsetminus E}\right)$ are arbitrarily close to $\mathcal{I}_0$ for t near zero, and arbitrarily close to $\mathcal{I}_1$ for t near one. Since $\mathcal{I}_0 \neq \mathcal{I}_1$, we can then choose, for all large j, a time t_{N_j} such that $f_{N_j}(t_{N_j})(S^2) \cap \left(\overline{\mathbb{C}P^2 \smallsetminus E}\right)$ is a fixed distance, say $\epsilon > 0$, from $\bigcup_{i=1}^{l} \overline{I}_i$. Passing to a subsequence, if necessary, we may then assume that the t_{N_j} converge to some $t_\infty \in [0,1]$ and that $f_{N_j}(t_{N_j})$ converges to a holomorphic building, $\mathbf{F}_\infty$. By the choice of the N_j, the images of the curves of $\mathbf{F}_\infty$ with target $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ intersect $\overline{\mathbb{C}P^2 \smallsetminus E}$ in a set whose distance from $\cup_{i=1}^{l} \overline{I}_i$ is at least ϵ. Hence, at least one of the curves of $\mathbf{F}_\infty$ with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ is not rigid and so has virtual index at least 2. This is the desired contradiction of Proposition 2.4.

$\square$

Remark 2.15. The above proof gives the following information on the images of the $f_N(t)$ which is a convenient way to apply Proposition 2.14. Let $K \subset (\mathbb{C}P^2 \smallsetminus E)^\infty_-$ be a compact subset with smooth boundary ∂K such that the limiting component of the $f_{N_j}(0)$ in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$, say F, intersects ∂K transversally. Define $D = F^{-1}(K)$ and $D_j(t) = f_{N_j}(t)^{-1}(K)$. Then given any $\epsilon > 0$ and sequence $t_j \in [0,1]$ there exists a j_0 such that if $j \geq j_0$ then there is a diffeomorphism $g_j : D \to D_j$ such that $f_{N_j}(t_j) \circ g_j$ is ϵ close to F in a C^∞-norm. (We can define this norm by fixing metrics on D and K.)

Indeed, if the $f_{N_j}(t_j)$ converge to a building with top level component identical to F then we can take the g_j to be a small perturbation of the maps $\sigma_{N_j}^{-1}$ implicit in the definition of convergence in Section 2.3.2. We notice that if $f_{N_j} \circ \sigma_{N_j}^{-1}$ is sufficiently C^∞ close to F and F intersects ∂K transversally then by the implicit function theorem $f_{N_j} \circ \sigma_{N_j}^{-1}$ can be slightly perturbed such that the preimage of ∂K coincides with that of F. On the other hand, if the $f_{N_j}(t_j)$ do not converge to a building with top level component identical to F then after taking a subsequence we will derive a contradiction to Proposition 2.14.

2.6 Holomorphic planes away from E

The key to the proof of Theorem 1.1 is the following existence result.

Theorem 2.16. *For any integer $d \geq 1$ and a suitable choice of almost-complex structure $J \in \mathcal{J}_E$, there exists a regular, finite energy holomorphic plane of degree d in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ whose negative end covers the periodic orbit γ_1 precisely $3d-1$ times.*

Here, as before, E is the (the image of the) ellipsoid $E(1/S, 1)$ and we assume that $S > \sqrt{3}d$.

Remark 2.17. Our proof of this theorem is quite lengthy. The equivalent result for embeddings of E into $S^2 \times S^2$ is that there exists a finite energy plane in $(S^2 \times S^2 \setminus E)^\infty_-$ whose negative end covers γ_1 precisely $2d+1$ times, and gluing a $2d+1$ times cover of the disk $\{z_2 = 0\} \cap E$ to the plane produces a homology class $(d, 1)$. This turns out to have a simpler proof. On the one hand, the fact that the corresponding Gromov-Witten invariant for holomorphic spheres in the class $(d, 1)$ through $2d+1$ points is 1 allows a proof as described in [9]. Alternatively, the fact that such spheres are all embedded allows an application of Embedded Contact Homology, see [14], [15]. The index formulas in ECH imply immediately that when we stretch the neck there can only be a single component in $(S^2 \times S^2 \setminus E)^\infty_-$, and it has a single negative end. Unfortunately degree d curves in $\mathbb{C}P^2$ are not embedded, and neither is the relevent Gromov-Witten invariant equal to 1. Thus we were led to follow the argument described below.

The rest of this section is devoted to the proof of Theorem 2.16. We begin by fixing points $p_1, \dots, p_{3d-1}$ in $E \subset \mathbb{C}P^2$ and an almost-complex structure $J \in \mathcal{J}_E$ as in Lemma 2.12. In particular, we assume that J agrees with the standard complex structure in $3d-1$ disjoint Darboux balls of radius slightly less than $1/S$ which are contained in E and centered around the points p_i. We also assume that J induces almost-complex structures on $(\mathbb{C}P^2 \setminus E)^\infty_-$, SE, and E^∞_+ which are regular for somewhere injective, finite energy curves of genus zero.

Consider the set of holomorphic buildings which occur as the limit of a sequence of curves representing classes in the spaces $\mathcal{M}_d(J^N, p_1, \dots, p_{3d-1})$. By Lemma 2.12 these limit buildings each have a single curve with image in $(\mathbb{C}P^2 \setminus E)^\infty_-$ and the negative ends of this curve cover γ_1 a total of $3d-1$ times. Let m be the maximum number of times a single negative end of one of these limit curves in $(\mathbb{C}P^2 \setminus E)^\infty_-$ covers γ_1. To prove Theorem 2.16 it suffices to show that $m = 3d-1$.

We will start by showing that $m > 1$. Arguing by contradiction, we will then assume that $m \in (1, 3d-1)$ and derive some consequences of this assumption on the curves of our limit buildings, expressed in the language of partitions. Ultimately, we derive a contradiction from these consequences by using them to detect a limit building whose curve with image in $(\mathbb{C}P^2 \setminus E)^\infty_-$ has a negative end which covers γ_1 at least $m+1$ times.

Proposition 2.18. *$m > 1$.*

Proof. Assume that this is false and that for any limit $\mathbf{F}$, as above, the unique curve F of $\mathbf{F}$ in $(\mathbb{C}P^2 \setminus E)^\infty_-$ has precisely $3d-1$ negative ends each of which covers γ_1 once. By Lemma 2.12 these limits each have precisely $3d-1$ planar curves with image in E^∞_+, and by Remark 2.11 each of these curves passes through exactly one of the point constraints. Asymptotic convergence to a cylinder over the Reeb orbit implies that we can find a closed subset $K \in (\mathbb{C}P^2 \setminus E)^\infty_-$ such that $F^{-1}(K)$ is topologically a sphere with $3d-1$ open disks removed.

For each N, we take a family of paths $p_i(t)$ in E with $p_i(0) = p_i$ and $p_i(1) = p$ for all i, where p is in E. We cannot guarantee that the families of point constraints $\{p_i(t), t \in [0,1]\}$ are generic in the sense of Proposition 2.13. Indeed, the moduli space of degree d curves passing through the $p_i(1) = p$ has high dimension. Nevertheless, we may choose the families $\{p_i(t), t < 1\}$ generically so that as in Proposition 2.13 there exist corresponding families of curves $f_N(t)$, for $0 \le t < 1$, which pass through the points $p_i(t)$ and satisfy $f_N(0) = f_N$. A limit may not exist as $t \to 1$. However by Gromov compactness there exists a sequence $t_i \to 1$ such that the curves $f_N(t_i)$ converge. The limit is a possibly nodal J^N-holomorphic sphere $g_N \colon \mathcal{S} \to \mathbb{C}P^2$ of degree d with $3d-1$-marked points $y_1, \dots, y_{3d-1}$ mapping to p.

Remark 2.19. If we think in terms of stable holomorphic maps from Riemann surfaces with marked points, the limiting curve g_N here could certainly include ghost bubble components mapping onto p. For example, in the case of degree $d = 1$ and points $p_1(t)$ and $p_2(t)$, after perhaps taking a subsequence curves $f(t) : (S^2, y_1, y_2) \to (\mathbb{C}P^2, p_1(t), p_2(t))$ in the homology class of a line will converge to a nodal curve consisting of a degree 1 curve through the point p and a ghost bubble containing the marked points y_1 and y_2. Which nontrivial curve arises in the limit depends upon the (complex) angle of approach of $p_1(t)$ and $p_2(t)$ to p in our subsequence. More precisely, there is a 2-parameter family of degree 1 curves in $\mathbb{C}P^2$ passing through p, parameterized by their tangent plane in $T_p\mathbb{C}P^2$, that is, by a $\mathbb{C}P^1$. If we fix $p_1(t) = p$ and choose $p_2(t)$ to converge when lifted to the blow-up of $\mathbb{C}P^2$ at the point p, then the tangency of the limit is determined by the limit of the $p_2(t)$ in the exceptional divisor.

We do not aim to specify the limit g_N precisely, but for our fixed N and any suitable sequence $t_i \to 1$ we make the following claim.

Claim 2.20. *The complement of* $g_N^{-1}(K)$ *does not have* $3d-1$ *separate components each containing a marked point. (Component here is meant topologically, it does not refer to components of a nodal curve.)*

We recall that by our choice of K and the convergence of $f_N(0)$ to $\mathbf{F}$ we have that for N sufficiently large the complement of $f_N(0)^{-1}(K)$ *does* have $3d-1$ components each containing a single marked point. Given Claim 2.20, by the convergence to g_N we can find a t_N in $(0,1)$ such that $f_N(t_N)$ also has the property that the complement of $f_N(t_N)^{-1}(K) \subset S^2$ does *not* have $3d-1$ components each containing a marked point. For convenience let us change our parameterizations and call $f_N(t_N)$ instead just $f_N(1)$. (In any case, there is a family of curves connecting it to $f_N(0)$.) Repeating this construction for each N, we get paths $f_N(t)$ which connect curves that pullback K to domains with different topology.

Now we make the following claim.

Claim 2.21. *For these families of curves,* $f_N(t)$ *with* $0 \leq t \leq 1$*, the conclusion of Proposition 2.14 still holds. That is, if the subsequences* $f_{N_j}(0)$ *and* $f_{N_j}(1)$ *both converge, then the curves with image in* $(\mathbb{C}P^2 \setminus E)^\infty_-$ *of the corresponding limiting holomorphic buildings, are identical.*

Proposition 2.18 follows immediately from the two claims. For if the limiting components in $(\mathbb{C}P^2 \setminus E)^\infty_-$ are identical, as they are according to Claim 2.21, then for large N_j the pullbacks $f_{N_j}(0)^{-1}(K)$ and $f_{N_j}(1)^{-1}(K)$ must have the same topology, but this contradicts Claim 2.20.

Proof of Claim 2.20. If the claim is false then there are $3d-1$ distinct components of $g_N^{-1}(E)$ each of which contains a marked point that is mapped by g_N to p. On the other hand, for each N there exists a (unique) J^N-holomorphic sphere D in $(\mathbb{C}P^2)^N$ which has degree one and passes through p and another fixed point $q \in (\mathbb{C}P^2)^N$. By positivity of intersection we then have $d = D \cdot (g_N(\mathcal{S})) \geq 3d-1 > d$, a contradiction.

Proof of Claim 2.21. We argue as in the Proposition 2.14, see also Remark 2.15. The new difficulty is that we cannot necessarily assume that the point constraints are regular. On the other hand the many components in E^∞_+ will be enough to ensure that curves outside of E cannot have strictly positive deformation index.

Assume, arguing by contradiction, that the limits of $f_{N_j}(0)$ and $f_{N_j}(1)$ in $(\mathbb{C}P^2 \setminus E)^\infty_-$ are distinct. Consider the compact subsets $\overline{I}_1, \ldots, \overline{I}_l$ of K

defined similarly to Proposition 2.14, i.e., each I_i is the intersection of K with of the image of a rigid curve in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$. (By compactness there are only finitely many such genus 0 curves of bounded degree.) Fixing an $\epsilon > 0$ sufficiently small, for j large we can choose a t_{N_j} such that $f_{N_j}(t_{N_j})(S^2) \cap K$ is at least a distance ϵ from $\bigcup_{i=1}^l \overline{I}_i$ and $f_{N_j}(t_{N_j})(S^2)$ still has $3d-1$ components in a compact neighborhood of E, each passing through a marked point. (We observe that this number of components cannot decrease without the component in K itself moving.) Passing to a subsequence, if necessary, we may then assume that the t_{N_j} converge to some $t_\infty \in [0,1]$ and that $f_{N_j}(t_{N_j})$ converges to a holomorphic building which has a nonrigid curve in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$, and which has at least $3d-1$ curves with image in E^∞_+. The curves in E^∞_+ have unconstrained deformation index at least 2 and therefore the total unconstrained index of limiting curves in E^∞_+ is at least $2(3d-1)$. On the other hand, degree d curves in $\mathbb{C}P^2$ have deformation index $6d-2$. So, as our almost-complex structure J induces a regular almost structure on $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$, any curves of this limit with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ must have index exactly 0. This is a contradiction since, by construction, there is a curve of the limit in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ which is not rigid.

□

In order to establish Theorem 2.16, that is, that $m = 3d-1$, we argue by contradiction. Given Proposition 2.18, this means that we assume $m \in (1, 3d-1)$. Let $\mathbf{F}$ be a limiting holomorphic building for a subsequence of the curves f_N with $[f_N] \in \mathcal{M}_d(J^N, p_1, \ldots, p_{3d-1})$, such that the unique curve of $\mathbf{F}$ with image in $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$ has a negative end which covers γ_1 exactly m times. We will use this particular limit $\mathbf{F}$ to derive our contradiction. Denote the level of $\mathbf{F}$ by k and its domain by $(\mathcal{S}, j)$. For simplicity, we will suppress the choice of a subsequence, and assume that the f_N converge to $\mathbf{F}$ in the sense of [2].

Partitions associated to F. The limiting building $\mathbf{F}$ can be used to define partitions of the set of point constraints $\{p_i\} = \{p_1, \ldots p_{3d-1}\}$ as follows. Let G_E be the collection of curves of $\mathbf{F}$ with images in either SE or E^∞_+ and denote by $\overline{G}_E$ the map to E formed by fitting together the compactifications of the curves of G_E, as described in Section 2.3.1. It follows from Lemma 2.12, that the domain of $\overline{G}_E$ is a union of discs. The intersections of the images of these discs with the point constraints determines a partition P of

$\{p_i\}$. In particular, we say that two points lie in the same block of P if they lie in the image of the same disc of the domain of $\overline{G}_E$.

Our assumption that $m < 3d-1$ implies that there are at least two discs in the domain of $\overline{G}_E$. So, we may assume that p_1 lies in the image of a disc whose end covers γ_1 a total of m times, and that p_2 lies in a different disc. Since $m > 1$, it follows from monotonicity and our choice of J, see Remark 2.11, that the disc through p_1 intersects at least one other point constraint, say p_3. In other words we can label the point constraints so that p_1 and p_3 lie in the same block of P and this block doesn't contain p_2.

Similarly, for any integer $r \in [0, k]$ we can define a partition P_r of the set $\{p_i\}$ by considering the maps to E constructed from the compactifications of the curves of $\mathbf{F}$ with level at most r, e.g., $P_k = P$. It follows from Lemma 2.12 that for $r > r'$ the partition $P_{r'}$ is a (possibly trivial) refinement of P_r. Hence p_1 and p_2 lie in different blocks of each P_r.

A distinguished level of F. Let r_0 in $[0, k]$ be the lowest level such that for all $r \leq k$ satisfying $r > r_0$, the curve of G_E at level r that corresponds to the block of P_r containing p_1 has exactly one positive and one negative end, each of which cover γ_1 exactly m times. Equivalently, r_0 can be defined as the smallest integer in $[0, k]$ such that the block of P_r which contains p_1 is the same for all $r \geq r_0$.

Throughout the proof of Theorem 2.16 the cases $r_0 = 0$ and $r_0 > 0$ will need to be dealt with separately. The latter case will require a significant amount of additional technical work. In preparation for this, we henceforth assume that the constraint points have been relabeled, if necessary, so that the point p_3 lies in a different block to p_1 for all partitions P_r with $r < r_0$. We also assume that the maps f_N converging to $\mathbf{F}$ have been chosen so that $f_N(0) = p_1$, $f_N(1) = p_3$ and $f_N(\infty) \in \Sigma_\infty$, where Σ_∞ is the line at infinity in $\mathbb{C}P^2$. For each N, this choice is not unique since our curves intersect the line at infinity in d points. However, we note that (as required later) such a choice can be made continuously over generic 1-parameter families of curves, $f_N(t)$.

A special curve of F of level r_0. We now identify a special curve of $\mathbf{F}$ of level r_0 in terms of its domain. To do this we consider the graphs $\mathrm{id} \times f_N \colon S^2 \to S^2 \times (\mathbb{C}P^2)^N$ of the curves f_N. For these maps, we have the following convergence statement.

Theorem 2.22. *(Theorem 10.6 of [2]) A subsequence of the maps* $\mathrm{id} \times f_N$ *converges to a holomorphic building (see Section* 2.3.1*).*

In this setting, the components of the limiting holomorphic building map to either $S^2 \times (\mathbb{C}P^2 \smallsetminus E)^\infty_-$, $S^2 \times SE$ or $S^2 \times E^\infty_+$. These are almost-complex manifolds with cylindrical ends, but now each end comes with Morse-Bott S^2-families of closed Reeb orbits.

Ignoring again the passage to a subsequence, let us assume that the maps $\mathrm{id} \times f_N$ converges to a holomorphic building with domain $(\mathcal{S}, j)$. Recalling the sequence of maps σ_N in the statement of the compactness theorem and looking at the S^2-components of the curves, it follows that the sequence of maps $\mathrm{id} \circ \sigma_N^{-1}$ converges to a continuous map, say $\mathfrak{z} \colon \mathcal{S} \to S^2$, which has removable singularities at the nodes of $\mathcal{S}$. For the $(\mathbb{C}P^2)^N$-components of the graphs, the sequence of maps $f_N \circ \sigma_N^{-1}$ converges to a holomorphic building, say $\mathbf{F}$, with the same domain $(\mathcal{S}, j)$ (at least if we allow trivial components).

The map $\mathfrak{z}$ is actually holomorphic of degree 1 since it is a limit of such maps. Hence, it is nonconstant on exactly one component of $\mathcal{S} \smallsetminus \{\text{nodes}\}$. Let $\mathcal{S}_f$ be the unique component of $(\mathcal{S}, j)$ on which $\mathfrak{z}$ is nonconstant, and denote the curve of $\mathbf{F}$ with this domain as f and its level by l.

Lemma 2.23. *The curve f maps into either SE or E^∞_+, that is, $l \leq k$. In fact, with p_3 chosen as above, f is the curve of level $l = r_0$ whose compactification lies in the component of $\bar{G}_E$ containing the points p_1 and p_3. If f maps into SE then the values of $\mathfrak{z}$ at the punctures of $\mathcal{S}_f$ include $\{0, 1, \infty\}$.*

Proof. If $r_0 = 0$ then there is a curve of $\mathbf{F}$ with image in E^∞_+ that passes through p_1 and p_3. This curve must be f since $\mathfrak{z}$ is nonconstant on its domain which includes both the values 0 and 1.

Now suppose that $r_0 > 0$ and consider the curve of level r_0 that corresponds to the block of P_{r_0} containing p_1. This curve has one positive puncture asymptotic to $\gamma_1^{(m)}$ and at least two negative punctures, one connected through a chain of curves to a curve through p_1, and another connected similarly to a curve through p_3 (these punctures are different by the choice of p_3). The positive puncture is connected through a chain of curves to a curve through Σ_∞. Now, $\mathfrak{z}$ takes the values 0, 1 and ∞ on the domains of the curves of $\mathbf{F}$ which pass through p_1, p_3 and Σ_∞, respectively. Therefore, as the function $\mathfrak{z}$ can be nonconstant on at most one of these three distinct chains of spheres, the value of $\mathfrak{z}$ at (at least) two of the punctures on our curve at level r_0 are different. Hence, $\mathfrak{z}$ is nonconstant on the domain of this

curve, which must therefore be f. It also follows that $\mathfrak{z}$ must be constant on the three chains of curves connected to the curves passing through p_1, p_3 and Σ_∞. Thus, we see that f has punctures at 0, 1 and ∞.

□

A first technical aside. In this aside we show that Theorem 2.22 applies equally well to the graphs of maps satisfying the holomorphic curve equation with respect to a domain dependent almost-complex structure, provided that the domain-dependence is carefully controlled near the neck region. To be precise, let $\tilde{f}_N \colon S^2 \to (\mathbb{C}P^2)^N$ be a sequence of maps which satisfy the holomorphic curve equation with respect to a domain dependent almost-complex structure $\tilde{J}^N$. That is, in coordinates $z = x + iy$ on S^2, we have

$$\frac{\partial \tilde{f}_N}{\partial y} = \tilde{J}^N(z, \tilde{f}_N(z)) \frac{\partial \tilde{f}_N}{\partial x}. \tag{12}$$

In our case, we will always be able to assume that in fact $\tilde{J}^N(z, \tilde{f}_N(z)) = \tilde{J}^N(\tilde{f}_N(z))$ is domain independent unless $\tilde{f}_N(z)$ lies in a fixed subset, say $\partial E \times [-1, 1]$, of the cylindrical portion of $(\mathbb{C}P^2)^N$.

Fix almost-complex structures $\bar{J}^N$ on $S^2 \times (\mathbb{C}P^2)^N$ such that for $z \in S^2$ and $p \in (\mathbb{C}P^2)^N$ we have $\bar{J}^N(z, p) = i \oplus \tilde{J}^N(z, p)$ where i is the standard complex structure on $T_z S^2$ and $\tilde{J}^N(z, p)$ is a complex structure on $T_p(\mathbb{C}P^2)^N$. (So for $p \notin \partial E \times [-1, 1]$ we have $\tilde{J}^N(z, p) = \tilde{J}^N(p)$.) Then the idea, going back to Corollary 1.5.E_1 of [6], is that $\tilde{f}_N$ satisfies the domain-dependent holomorphic curve equation (12) if and only if its graph $\mathrm{id} \times \tilde{f}_N \colon S^2 \to S^2 \times (\mathbb{C}P^2)^N$ is $\bar{J}^N$-holomorphic.

As $N \to \infty$ we require the $\bar{J}^N$ to converge to an almost-complex structure $\bar{J} = i \oplus J$. The convergence is uniform in the smooth topology on E and $\mathbb{C}P^2 \setminus E$. In the remaining portion of $(\mathbb{C}P^2)^N$, $\partial E \times [-N, N]$, at least up to translation, we require our $\bar{J}^N$ to be independent of both z and N and translation invariant on $\partial E \times ((-N, -1] \cup [1, N))$, and uniformly convergent on $\partial E \times [-N, N]$.

In this setting, as $N \to \infty$ the complex structure is now stretched as before by gluing copies of $\partial E \times [-N, N]$ to $\mathbb{C}P^2 \setminus E$ where the almost-complex structure is extended to be translationally invariant outside of $\partial E \times [-1, 1]$. This is equivalent to stretching along two copies of $S^2 \times \partial E$ (or one if we have translation invariance on all of $\partial E \times (-\infty, \infty)$). Then the compactness theorem of [2] holds as before

Theorem 2.24. *([2]) A subsequence of the maps* $\mathrm{id} \times \tilde{f}_N$ *converges to a holomorphic building.*

The curves of the limiting building map into one of $S^2 \times (\mathbb{C}P^2 \smallsetminus E)^\infty_-$, $S^2 \times SE$ or $S^2 \times E^\infty_+$, where the almost-complex structure on $S^2 \times SE$ is now either translation invariant or else translation invariant outside of a compact set. Moreover, the curves which map into the copy of $S^2 \times SE$ whose almost-complex structure is not translationally invariant, all have the same level. The sequence of maps $\mathrm{id} \circ \sigma_N^{-1}$ converges to a continuous map $\tilde{\mathfrak{z}} \colon \tilde{\mathcal{S}} \to S^2$, and the sequence of maps $\tilde{f}_N \circ \sigma_N^{-1}$ converges to a holomorphic building, say $\tilde{\mathbf{F}}$, with the same domain $(\tilde{\mathcal{S}}, j)$ (again allowing for trivial components). In local coordinates, the curves of $\tilde{\mathbf{F}}$ now satisfy equation (12) with $\tilde{J}^N$ replaced by its limit J, and z replaced by $\tilde{\mathfrak{z}}(x, y)$.

Partitions associated to the f_N. We now define partitions of the $\{p_i\}$ which are intrinsically associated to the sequence of curves f_N converging to $\mathbf{F}$ (rather than referring to the limiting building $\mathbf{F}$, itself). This can be done by selecting a sequence of subsets $A_N \subset \partial E \times (-N, N) \subset (\mathbb{C}P^2)^N$.

Definition 2.25. For sequences f_N and A_N as above, the partition $P(A_N, f_N)$ of the set $\{p_i\}$ is defined by saying that two points p_j and p_k lie in the same block if for all large N there exists a continuous path in $S^2 \smallsetminus f_N^{-1}(A_N)$ which starts at y_j and ends at y_k.

Here the $y_i \in f_N^{-1}(p_i)$ are as in the definition of $\mathcal{M}_d(J, p_1, \ldots, p_M)$ from Section 2.1.

As we now describe, the partition P_{r_0} (in fact any of the P_r) corresponds to certain partitions defined as in Definition 2.25. For the case $r_0 > 0$, consider the sequence of neighborhoods of the form

$$A_N = \partial E \times (\mathbf{s} - s_N^{r_0} - 1, \mathbf{s} - s_N^{r_0} + 1),$$

where $\mathbf{s} \in \mathbb{R}$ and the $s_N^{r_0}$ are the shifts that determine the convergence of the f_N at level r_0 (see Section 2.3.2). For large enough N, each A_N lies in the cylindrical portion $\partial E \times (-N, N)$ of $(\mathbb{C}P^2)^N$. The sets

$$\psi^{s_N^{r_0}} \left(f_N(S^2) \cap A_N \right)$$

converge to a subset of $v_{r_0}(\mathcal{S}_{r_0})$, the image of the level r_0 curves of $\mathbf{F}$. In particular, modulo translations, the sets A_N can be identified with the set

$A = \partial E \times (\mathbf{s} - 1, \mathbf{s} + 1)$ in the copy of SE which acts as the target of the level r_0 curves of $\mathbf{F}$. Choosing $\mathbf{s}$ to be sufficiently large, we may assume that the sets $\sigma_N(f_N^{-1}(A_N))$ converge to a collection of annuli, one in each component of $\mathcal{S}_{r_0}$, which are arbitrarily close to the positive asymptotic puncture in each component. That is, the components of the complements of these annuli that contain the positive punctures are precisely punctured disks. The convergence of f_N to $\mathbf{F}$ implies that $P_{r_0} = P(A_N, f_N)$.

If $r_0 = 0$, then $s_N^{r_0} = 0$ and we recover P_{r_0}, as above, by simply setting $A_N = \partial E \times (\mathbf{s} - 1, \mathbf{s} + 1)$ for sufficiently large $\mathbf{s} > 0$.

Another building realizing m. We now observe that there exists another limiting building $\mathbf{F}'$ whose curve in $(\mathbb{C}P^2 \setminus E)_-^\infty$ agrees with that of $\mathbf{F}$, but which determines a different family of partitions of $\{p_1, \dots p_{3d-1}\}$. To see this, recalling Section 2.5, consider the space $\mathcal{N}_N$ determined by a family of paths $p_i(t)$ in E which switches p_1 and p_2 and leaves the other points fixed. More precisely, suppose that $p_i(0) = p_i$ for all i, $p_1(1) = p_2$, $p_2(1) = p_1$, and $p_i(1) = p_i$ for all $i > 2$. Let $f_N(t)$ be a 1-parameter family of curves representing a component of $\mathcal{N}_N$ such that $[f_N(0)] = [f_N]$. Set $f'_N = f_N(1)$. Passing to subsequences we may assume that the f'_N also converge to some $\mathbf{F}'$. By Proposition 2.14, the curves of $\mathbf{F}$ and $\mathbf{F}'$ with image in $(\mathbb{C}P^2 \setminus E)_-^\infty$ are the same. On the other hand, by our choice of the paths $p_i(t)$, it follows from Proposition 2.13 that the points p_1 and p_3 are in different blocks of the partition P' of $\{p_1, \dots p_{3d-1}\}$ which corresponds to $\mathbf{F}'$. Hence $P \neq P'$.

More importantly, we observe the following.

Lemma 2.26. *For $A_N = \partial E \times (\mathbf{s} - s_N^{r_0} - 1, \mathbf{s} - s_N^{r_0} + 1)$, the partition $P(A_N, f'_N)$ is not equal to $P(A_N, f_N)$. In particular, the points p_1 and p_3 lie in different blocks of $P(A_N, f'_N)$.*

Remark 2.27. Here we are using the same subsets A_N to define both partitions (even though the subsets were originally defined in terms of f_N, not f'_N).

Proof. To see this, we argue by contradiction. Define $K \subset (\mathbb{C}P^2 \setminus E)_-^\infty$ such that the level $k + 1$ component F of $\mathbf{F}$ intersects ∂K transversally and restricted to the preimage of $(\mathbb{C}P^2 \setminus E)_-^\infty \setminus K$ the map F is, up to parameterization, C^∞ close to trivial holomorphic cylinders over the Reeb orbit. Assume there exists a path $\gamma = \gamma(1)$ in $S^2 \setminus (f'_N)^{-1}(A_N)$ which connects ${f'_N}^{-1}(p_1)$ and ${f'_N}^{-1}(p_3)$. We observe that in particular γ has image

lying in $S^2 \setminus (f'_N)^{-1}(K)$ since in any $(\mathbb{C}P^2)^N$ the sets A_N separate E from $\mathbb{C}P^2 \setminus E$. Let $f_N(t)$ be the family of curves used to define $\mathbf{F}'$ above. By Proposition 2.14, see also Remark 2.15, for large N the sets $f_N(t)^{-1}(K)$ have the same topology relative to the marked points. Therefore the path $\gamma(1)$ can be included in a homotopy of paths $\gamma(t)$ connecting $f_N^{-1}(p_2(t))$ to $f_N^{-1}(p_3)$ within $S^2 \setminus f_N(t)^{-1}(K)$. But then $\gamma(0)$ connects $f_N^{-1}(p_2)$ and $f_N^{-1}(p_3)$ in $S^2 \setminus f_N^{-1}(K)$, which contradicts the fact that p_2 and p_3 are in different blocks of P.

$\square$

Strategy of the proof of Theorem 2.16. For the sake of clarity, we briefly outline the plan for the remainder of the proof. At present, we have two convergent sequences of curves, f_N and f'_N, whose limiting buildings have the same curve with image in $(\mathbb{C}P^2 \setminus E)^\infty_-$, and this curve has a negative end that covers γ_1 exactly m-times, the maximum covering number among all such curves in such limits. Moreover, by construction, the corresponding partitions $P(A_N, f_N)$ and $P(A_N, f'_N)$ of the constraint points differ, in that p_1 and p_3 lie in the same block of the former partition, and in different blocks of the latter.

We now forget the constraint point p_2. The sequences of maps f_N and f'_N, and the sets A_N, still induce different partitions of the remaining point constaints $\{p_1, p_3, \ldots, p_{3d-1}\}$. By Lemma 2.2, for each N the maps f_N and f'_N can be connected through a path of maps $f_N(t) : S^2 \to (\mathbb{C}P^2)^N$, for $0 \le t \le 1$, such that $[f_N(t)] \in \mathcal{M}_d(J^N, p_1, p_3, \ldots, p_{3d-1})$. For all N and t, the subset $f_N(t)^{-1}(A_N) \subset S^2$ is disjoint from $f_N(t)^{-1}(\{p_i\}_{i \neq 2})$. Since the partitions of $\{p_i\}_{i\neq 2}$ determined by components of the complements of the $f_N(t)^{-1}(A_N)$ are different for $t = 0$ and $t = 1$, we conclude that the topology of $f_N(t)^{-1}(A_N)$ must change with t. This change will be the eventual source of our contradiction to the assumption that m is both maximal and less than $3d - 1$.

To exploit this change we must first arrange that f, the curve of $\mathbf{F}$ of level r_0 that corresponds to the block of the partition containing p_1, is isolated in SE or E^∞_+. (Recall that level r_0 curves include the limits of the curves through the A_N.) Proposition 2.4 implies that f must have deformation index 0. However, in the case $r_0 > 0$ this does not imply that f is isolated. Indeed, multiple covers of cylinders in SE with positive and negative ends covering γ_1 an equal number of times have deformation index zero but are never isolated as they admit a free $\mathbb{C}^*$-action. The underlying

trivial cylinders themselves also have deformation index zero and therefore cannot be removed by generic perturbations of the almost-complex structure on SE. To avoid this possibility we use almost-complex structures which are domain-dependent on the preimages of the A_N to obtain a rigid perturbation of f. This procedure is the subject of the forthcoming technical aside. (The fact that the resulting rigid curve appears in a limiting holomorphic building corresponding to a sequence of holomorphic spheres, follows from Theorem 2.24 of our first technical aside.)

Now, by compactness, there are only finitely many images of isolated curves (with energy bounded by a constant) in a copy of SE or E_+^∞. It follows that for large N and some $t_N \in (0,1)$ that $f_N(t_N)(S^2) \cap A_N$ must lie a fixed distance (in a sense to be made precise) from the images of isolated curves intersected with A_N (thinking now of A_N as a subset of an SE). Otherwise we could not have the change of topology detected above. The limiting building of a convergent subsequence of the $f_N(t_N)$ therefore includes a curve which lies a fixed distance from all isolated curves and hence is not isolated itself. Therefore it will have positive deformation index, and adding back the point constraint p_2 to this curve we still have a curve with nonnegative constrained deformation index. But the curve already corresponds to the block of the partition $P(A_N, f_N(t_N))$ that includes p_1 and has m elements. Adding p_2 it now generates a partition containing a block of size $m+1$. This will give a contradiction to the hypothesis that m is maximal.

A second technical aside. To complete the proof of Theorem 2.16 we will need the distinguished curve f discussed in Lemma 2.23 to be rigid, that is, isolated and regular, and in particular of deformation index zero. This is automatic if f maps to E_+^∞ (i.e., $r_0 = 0$) and the constraint points are in generic position. Let us assume then that $r_0 > 0$ in which case f maps to SE and is a multiply covered cylinder with one positive and at least two negative punctures. (It is not a trivial cylinder by the definition of r_0 above.) To achieve rigidity for f we will perturb it so that the resulting curve $\tilde{f}$ is somewhere injective, and is a regular solution of a Cauchy-Riemann type equation with some domain dependence. This process is carried out carefully in the following paragraphs, and is possible precisely because f is the unique curve of $\mathbf{F}$ on which the map $\mathfrak{z}$ is nonconstant. The curve $\tilde{f}$ will have the same asymptotic behavior as f, and the other curves of $\mathbf{F}$ will be unchanged by this procedure. In particular, the new limiting building will still be a limit of holmorphic spheres (holomorphic with respect to a domain

dependent almost-complex structure) in the sense of Theorem 2.24. It will also have the same curve of level $k+1$ as $\mathbf{F}$ (that is, the same curve mapping to $(\mathbb{C}P^2 \smallsetminus E)^\infty_-$) with a negative end covering the orbit γ_1 precisely m times.

Let $A = \partial E \times (\mathbf{s} - 1, \mathbf{s} + 1) \subset SE$. As above, up to translation we identify A with the sets $\psi^{s_N^{r_0}}(A_N)$ and with the subset of the copy of SE that corresponds to the target of the level r_0 curves of $\mathbf{F}$. For large $\mathbf{s}$ the set $f^{-1}(A)$ is arbitrarily close to the positive puncture in the domain of f. By Lemma 2.23, f is asymptotically cylindrical to $\gamma_1^{(m)}$ at this puncture. Since $\mathfrak{z}\colon \mathcal{S} \to S^2$ is a diffeomorphism on the domain of f, we will use $\mathfrak{z}$ to identify this domain with its image in S^2, whose points will be denoted by z.

Let $\tilde{f}$ be a small perturbation of f on the preimage of A such that $\tilde{f} = f$ near the boundary of $f^{-1}(A)$ but $\tilde{f}$ is not a multiple cover. We can then define a domain dependent almost-complex structure $\tilde{J}$ on $\tilde{f}^{-1}(A) \times A$, (that is, a map $\tilde{J} : (z,p) \mapsto \mathrm{Aut}(T_pSE)$ defined for $z \in \tilde{f}^{-1}(A)$ and $p \in A$) such that $\tilde{f}$ becomes $\tilde{J}$-holomorphic. In other words, for local coordinates $z = x + iy$ on $\tilde{f}^{-1}(A) \subset S^2$, as in equation (12) we have

$$\frac{\partial \tilde{f}}{\partial y} = \tilde{J}(z, \tilde{f}(z)) \frac{\partial \tilde{f}}{\partial x}.$$

This is possible since on intersecting branches of $\tilde{f}$ the z coordinates are necessarily different. We extend $\tilde{f}$ such that $\tilde{f} = f$ outside of A, and extend $\tilde{J}$ such that $\tilde{J}(z,p) = J(p)$ for $p \notin A$. In particular $\tilde{J}(z,p) = J(p)$ if z is close to a puncture, for example if z is close to 0, 1 or ∞. Since $\mathfrak{z}$ is constant on the domains of all other curves of our limiting building $\mathbf{F}$ they can also be viewed as solutions of the $\tilde{J}$-holomorphic equation, which are unaffected by the domain dependence.

The $\tilde{J}$-holomorphic curve $\tilde{f}$ has the same asymptotic limits as f and thus still has virtual deformation index zero, this time in the space of parameterized spheres whose set of punctures includes the points 0, 1 and ∞. We note that the reparameterization group $PSL(2, \mathbb{C})$ does not act on parameterized holomorphic maps, so to remain in a moduli space of deformation index zero we need to impose the three point constraints. However, $\tilde{f}$ is no longer multiply covered and so, for a generic choice of $\tilde{J}$, it will be rigid and hence isolated from other maps of the sphere to SE which satisfy the new Cauchy-Riemann equation, and have bounded energy, deformation index equal to zero and punctures at 0, 1 and ∞. We recall from Lemma 2.23 that limits of parameterized holomorphic curves on which $\mathfrak{z}$ is nonconstant

do have punctures at 0, 1 and ∞, thus $\tilde{f}$ is isolated from other such limits. Let $\tilde{\mathbf{F}}$ be the holomorphic building obtained from $\mathbf{F}$ by replacing f by $\tilde{f}$.

By the nature of their convergence as parameterized maps to the original f and J, both f_N and J^N can be perturbed analogously to obtain a sequence of curves $\tilde{f}_N$ which converge to $\tilde{f}$ in the sense of Theorem 2.24 and are holomorphic with respect to a sequence of domain dependent almost-complex structures $\tilde{J}^N$ which converge to $\tilde{J}$ on A. More precisely, for $f_N(z) \notin \psi^{-s_N^{r_0}}(A)$ we can set $\tilde{f}_N(z) = f_N(z)$ and $\tilde{J}^N(z, f_N(z)) = J^N(f_N(z))$, and for $z \in f_N^{-1}(\psi^{-s_N^{r_0}}(A))$ we perturb $\tilde{f}_N$ to approach $\tilde{f}$ and define $\tilde{J}^N$ to approach $\tilde{J}$ and such that the $\tilde{f}_N$ are $\tilde{J}^N$-holomorphic. Note that this can be done to give smooth (domain dependent) almost-complex structures $\tilde{J}^N$.

Remark 2.28. A concrete way to do this would be to define $\tilde{f}$ by adding to f a multi-valued section g of the normal bundle to the image of f, where all branches of the section have support in A. For N large the images of the f_N, up to translation, are themselves multi-valued sections of this same normal bundle and can therefore be perturbed by adding the same g once we identify the relevent branches to ensure convergence.

The curves f'_N which converge to the limiting building $\mathbf{F}'$ can now be obtained from the $\tilde{f}_N$ as before (by following paths of points $p_i(t)$) but now using the domain dependent almost-complex structures $\tilde{J}^N$ (see the first technical aside). As we work now with parameterized curves, for the indices to correspond we must fix parameterizations as above so that the $\tilde{f}_N(t)$ satisfy $\tilde{f}_N(t)(0) = p_1(t)$ $\tilde{f}_N(t)(1) = p_3(t)$ and $\tilde{f}_N(t)(\infty) \in \Sigma_\infty$. By Proposition 2.14, the limit $\mathbf{F}'$ of the f'_N will still have the same curve as $\tilde{\mathbf{F}}$ in $(\mathbb{C}P^2 \setminus E)_-^\infty$.

Back to the proof of Theorem 2.16. To simplify the notation we now drop the tildes from the perturbed versions of J, f, and $\mathbf{F}$ defined in our second technical aside. We just retain the fact that the level r_0 curve f of $\mathbf{F}$ which corresponds to the block of P_{r_0} containing p_1, is now regular, isolated, and has deformation index zero.

As described previously, for sufficiently large N the components of $f_N^{-1}(A_N)$ and $(f'_N)^{-1}(A_N)$ determine partitions $P(A_N, f_N)$ and $P(A_N, f'_N)$ of the set $\{p_1, p_2, \ldots, p_{3d-1}\}$. We also recall (see Lemma 2.26 and Remark 2.27) that these partitions are distinct since p_1 and p_3 are in the same block of $P(A_N, f_N)$ but are in different blocks of $P(A_N, f'_N)$. Hence, if we forget the point p_2, then $P(A_N, f_N)$ and $P(A_N, f'_N)$ still determine distinct partitions, say P and P', of $\{p_1, p_3, \ldots, p_{3d-1}\}$.

By Proposition 2.2, we can connect f_N and f'_N by a continuous path of holomorphic curves $f_N(t)$ through the points $p_1, p_3, \ldots, p_{3d-1}$. We may assume that the $f_N(t)$ are all parameterized such that 0, 1 and ∞ map to p_1, p_3 and Σ_∞ as before. When N is large enough, the components of $(f_N(t))^{-1}(A_N)$ determine a family of partitions $P(t)$ of $\{p_1, p_3, \ldots, p_{3d-1}\}$ such that $P(0) = P$ and $P(1) = P'$. Let $c_N(t)$ be the restriction of $f_N(t)$ to the component of $(f_N(t))^{-1}(A_N)$ which corresponds to the block of $P(t)$ that contains p_1. Since the partitions $P(0)$ and $P(1)$ have different blocks containing p_1, the $c_N(t)$ must vary nontrivially with t. In fact, the topology of the domain of the $c_N(t)$ must change for some t. We now use this transition to obtain a contradiction to the assumption that $m < 3d - 1$.

Henceforth, we only consider values of N which are large enough for the maps $c_N(t)$ to be defined as above. For values of t near 0, each $c_N(t)$ is defined on an annulus whose boundary components are mapped to different components of ∂A_N. If we make an identification between their domains and the standard annulus $[-1, 1] \times S^1$, then these $c_N(t)$ define elements in the space

$$\left\{a \in C^\infty([-1, 1] \times S^1, A_N) \mid a(\{\pm 1\} \times S^1) \subset \{s_N \pm 1\} \times \partial E\right\}.$$

Consider the topology on this space determined by the distance function

$$\operatorname{dist}(a_1, a_2) = \inf \|a_1 \circ h - a_2\|_{C^\infty},$$

where the infimum is over smooth diffeomorphisms h of $[-1, 1] \times S^1$. Note that this also determines a well-defined (yet degenerate!) distance function on the maps $c_N(t)$, for values of t near 0, since it is independent of how we identify their domains with the standard annulus.

As described above, for some t the domain of $c_N(t)$ must no longer be an annulus. Hence, for all sufficiently small $\epsilon \in (0, 1]$ there is a minimal $t_N(\epsilon) \in (0, 1)$ such that the distance between $c_N(t_N)$ and $c_N(0)$ is at least ϵ. (In order for the topology of its domain to change, the derivative of $c_N(t)$ at the boundary must at some time be tangent to ∂A_N. If we assume that $c_N(0)$ is transverse to this boundary then at that time we can bound below the distance between $c_N(t)$ and $c_N(0)$.) Let us fix such an $\epsilon \in (0, 1]$ and set $t_N = t_N(\epsilon)$.

Taking a convergent subsequence of the $f_N(t_N)$ as $N \to \infty$, we obtain a holomorphic building $\mathbf{F}''$. The domain of the building again comes with a

map $\mathfrak{z}''$ to S^2 which is constant on all but one component of the domain and extends continuously across the nodes.

Some care must be taken with the meaning of the deformation index of the components of $\mathbf{F}''$. According to Lemma 2.23, the component of a limit on which $\mathfrak{z}''$ is nonconstant has punctures at $\mathfrak{z}'' = 0, 1, \infty$. Therefore when we compute the index of the curve on whose domain $\mathfrak{z}''$ is nonconstant, we work in a moduli space consisting only of curves with punctures at $\mathfrak{z}'' = 0, 1, \infty$. (The value of $\mathfrak{z}''$ at other punctures is allowed to vary.) For a building to arise as a limit it is necessary that the coordinate function be continuous. Therefore, once $\mathfrak{z}''$ is given on the nonconstant component it is determined on all other components (since the domain $\mathcal{S}$ is connected). The virtual indices of these other components (where there is a $PSL(2, \mathbb{C})$ action) are defined as before. With these conventions the index formulas for components of $\mathbf{F}''$ are given by the same formulas as before. In particular, the indices of all component curves of $\mathbf{F}''$ must sum to 2, since this is the index of curves in the original moduli space. (We have forgotten one point constraint.) Hence, in the generic case when all components have nonnegative index, at most one curve of $\mathbf{F}''$ has index 2.

We observe here that by our choice of t_N, a curve of $\mathbf{F}''$ in $(\mathbb{C}P^2 \setminus E)^\infty_-$ has a negative end which connects to a component in $E^\infty_+ \cup SE$ that passes through m fixed points. In particular, for $t \le t_N$, the domain of $c_N(t)$ is still an annulus and therefore still bounds a disc in the domain of $f_N(t)$ which intersects the same m marked points as in the block of P containing p_1.

Let f'' be the curve of $\mathbf{F}''$ with image in SE or E^∞_+ that includes the limits of the $c_N(t_N)$. (As this copy of SE, or E^∞_+, contains A, the complex structure is not translation invariant, so f'' is in fact well defined as a map to SE or E^∞_+, not just up to translation.) As $\epsilon > 0$, we have $f'' \neq f$. Nevertheless, we can assume that f'' is close enough to f that it's tangent planes cannot be holomorphic for a domain independent almost-complex structure (for example, because there may be self-intersection points of negative index), in other words f'' will be the unique component of $\mathbf{F}''$ on which the coordinate function $\mathfrak{z}''$ is nonconstant.

When $r_0 = 0$, we define $\mathcal{C}_{r_0}(e, d)$ to be the set of holomorphic maps of genus 0 into E^∞_+ which have index zero, energy less than e and the following third property: the preimage of A under the map has an annular component so that the restriction of the map to this component lies within distance d of the $c_N(0)$ (as defined above) for all N large. For the case $r_0 > 0$, we define $\mathcal{C}_{r_0}(e, d)$ to be the the space of genus 0 maps into SE which have punctures

at $\{0, 1, \infty\}$, are holomorphic with respect to the domain dependent almost-complex structure, and have the same three additional properties. Since we have arranged the $c_N(0)$ not to be multiply covered in both cases, we can choose d so that the third property above implies that the curves in $\mathcal{C}_{r_0}(e, d)$ are somewhere injective. Thus, $\mathcal{C}_{r_0}(e, d)$ is a finite set since it is compact and has dimension zero. For a suitably choice of $e > 0$ (and any choice of $d > 0$), we may assume that $\mathcal{C}_{r_0}(e, d)$ contains the curve f. For any curve in $\mathcal{C}_{r_0}(e, d)$ which does not coincide with f, its restriction to the annular components in the preimage of A all lie more than some fixed distance $d_1 < d$ away from the corresponding restriction of f. Choosing the parameter ϵ in the definition of f'' above to be smaller than d_1, we conclude that f'' is not in $\mathcal{C}_{r_0}(e, d)$. However, f'' satisfies all of the defining criteria for $\mathcal{C}_{r_0}(e, d)$ except that of having deformation index zero. Thus its deformation index is nonzero, f'' is not rigid, and $\mathrm{index}(f'') = 2$.

Remark 2.29. The sequence converging to $\mathbf{F}''$ seems to already give a contradiction as there is a single component in G_E now with (unconstrained) deformation index $2(m + 1)$. However we must be a little careful. The number m was defined as a maximum only over limits of curves passing through the original points $\{p_1, \ldots, p_{3d-1}\}$; in the situation at hand it may not be clear even that the domain of our relevent component of G_E is a topological disk. Thus the additional argument supplied below is perhaps necessary.

We now introduce a point $p_2^N(0)$ to the image of $c_N(t_N)$. In the limit we may assume that these points converge to a point p'' in the image of f'' which lies a distance ϵ from all rigid curves. Then in the moduli space of curves passing through $3d-1$ points (when we include $p_2^N(0)$ or p'') the curves $f_N(t_N)$ and the limiting curve f'' can be assumed to be rigid. There is now a component of each $f_N(t_N)$ restricted to $(f_N(t_N))^{-1}(E \cup (\partial E \times [-N, N]))$ passing through $m + 1$ points. To finally derive a contradiction we must find a limit through the original set of points $\{p_1, \ldots, p_{3d-1}\}$. To do this, we proceed as follows.

We extend $p_2^N(0)$ to a path $p_2^N(t) \in (\mathbb{C}P^2)^N$ such that $p_2^N(1) = p_2$. This can be done generically for each N and to satisfy the following conditions:

(i). for N large all $p_2^N(t)$ remain a bounded distance from the images of rigid curves in A_N (when identified with A);

(ii). for N large all $p_2^N(t)$ remain a bounded distance from the intersection of rigid curves through a subset of the $\{p_1, p_3, \ldots, p_{3d-1}\}$ with E;

(iii). for N large all $p_2^N(t)$ remain a bounded distance from $\gamma_1 \times [-N, N]$ in $\partial E \times [-N, N]$.

The sets above that the $p_2^N(t)$ must avoid are all of codimension 2 and so such paths $p_2^N(t)$ do exist. We can assume by exponential convergence that rigid curves in the copy of SE containing A are all very close to the cylinders $\gamma_1 \times \mathbb{R}$ outside of A (replacing A by a longer cylinder if necessary).

There are corresponding families of curves $f_N(t)$ for $0 \le t \le 1$ through $p_1, p_2^N(t), p_3, \ldots, p_{3d-1}$ connecting $f_N(t_N)$ to another curve h_N that passes through $p_1, p_2, \ldots, p_{3d-1}$.

Claim 2.30. *Let $s_N \in [0, 1]$ be a sequence and suppose that the sequence of curves $f_N(s_N)$ converges in the sense of Section* 2.3.1*. Then the limiting holomorphic building has the same components as $\mathbf{F}''$ in $(\mathbb{C}P^2 \setminus E)^\infty_-$.*

Claim 2.30 implies that for the 1-parameter family of curves connecting $f_N(t_N)$ and h_N, the intersections of the images of this family with any fixed compact subset $K \subset (\mathbb{C}P^2 \setminus E)^\infty_-$ (which can be canonically identified with a subset of each $(\mathbb{C}P^2)^N$ for N large) are approximately constant when N is large, see Remark 2.15. Hence, the components of the intersection of these curves with the complement of K determine identical partitions of the sets $\{p_1, p_2(t), \ldots, p_{3d-1}\}$. For N sufficiently large, it follows that the image of the curve h_N has a component in $(\mathbb{C}P^2)^N \setminus K$ passing through $m + 1$ points. Looking at the limit of the h_N and choosing K such that our limits are approximately cylindrical on $(\mathbb{C}P^2 \setminus E)^\infty_- \setminus K$, we find a component in $E^\infty_+ \cup SE$ that passes through $m + 1$ disjoint balls, implying that it must cover γ_1 at least $m + 1$ times. This contradicts the definition of m as the maximum such covering number.

Proof of Claim 2.30. This follows Proposition 2.14 closely. We just need to ensure that in any limit of $f_N(s_N)$ the sum of the unconstrained indices of the components in $E^\infty_+ \cup SE$ is at least $2(3d - 1)$.

The components intersecting $\{p_1, p_3, \ldots, p_{3d-1}\}$ have indices summing to at least $2(3d - 2)$. Suppose that, following the notation of Section 2.3.2, $\sigma_N(f_N(s_N)^{-1}(p_2^N(s_N)))$ converges to a point z in the limiting building. Then by our choice of $p_2^N(s_N)$ (avoiding cylinders over Reeb orbits, condition (iii)) the point z is not a node. If z lies in a component mapping to E^∞_+ which passes through k other constraint points, then by condition (ii) the limiting component has index at least $2k + 2$ and our sum is as required. If z lies in a component mapping to the copy of SE containing A, then by condition

(i) this component has index at least 2. Recall from Lemma 2.9 that any curves of deformation index zero in a copy of SE with its translation invariant almost-complex structure are multiple covers of the trivial cylinder over γ_1. Thus by condition (iii) if z lies in a component mapping to a copy of SE with its translation invariant almost-complex structure then again the component has index at least 2 and in all cases we can conclude as required.

3 The proof of Theorem 1.2

Let (M, ω) be the product manifold $\mathbb{C}P^2 \times \mathbb{C}^{n-2}$ equipped with the split symplectic form $R^2\omega_{FS} \oplus \omega_0$. When convenient, we will equip the $\mathbb{C}^{n-2}$–factor of M with complex coordinates $z_3, \ldots, z_n$. We will also consider the $(n-2)$-dimensional torus, $\mathbb{T}^{n-2}$, acting on $\mathbb{C}^{n-2}$ in the standard way by rotations in the z_j–directions.

Let $E(a_1, a_2, \ldots, a_n)$ be the ellipsoid

$$\left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=2}^{n} \frac{|z_i|^2}{a_i^2} \leq 1 \right\}$$

Suppose that for any $S > 0$ there exists a symplectic embedding

$$\phi(S) : E(1, S, \ldots, S) \hookrightarrow M.$$

To prove Theorem 1.2 we must show that this implies that $R \geq \sqrt{3}$.

Fix an integer $d \geq 1$, and a positive real number S such that S^2 is irrational and $S^2 > d(3d-1)$. Set $\phi = \phi(S)$. For $t \in [1/S, 1]$, let ϕ_t be a smooth family of symplectic embeddings

$$\phi_t : E_t = E(t, tS, \ldots, tS) \hookrightarrow M$$

such that:

- for t in some neighborhood of 1 the embeddings ϕ_t are just the restrictions of ϕ to E_t;
- $\phi_{1/S}$ coincides with the inclusion of $E(1/S, 1, \ldots, 1)$ into $E(1,1) \times \mathbb{C}^{n-2} \subset \mathbb{C}P^2(R) \times \mathbb{C}^{n-2} = M$.

In particular, the image of $\phi_{1/S}$ is chosen to be invariant under the action of $\mathbb{T}^{n-2}$ on M. In what follows, we will identify E_t with its image $\phi_t(E_t)$.

The standard contact form on ∂E_t has n simple closed Reeb orbits, whose images correspond to the intersections $\partial E_t \cap \{z_j = 0; j \neq i\}$ for $i = 1, \dots, n$. For simplicity, we will denote the closed Reeb orbit on ∂E_t with the smallest action by γ_1, i.e., we suppress the dependence on t. The action of γ_1 is πt^2 and the Conley-Zehnder index of its r-fold cover, $\gamma_1^{(r)}$, is given by

$$\mu(\gamma_1^{(r)}) = 2r + (n-1)\left(2\left\lfloor \frac{r}{S^2} \right\rfloor + 1\right).$$

Let $\mathcal{J}_t = \mathcal{J}_{E_t}$ be the space of smooth almost-complex structures on M which are compatible with the symplectic form on M and are also compatible with E_t in the sense of Section 2.3. As described in that same section, $J_t \in \mathcal{J}_t$ determines an almost-complex structure on $(M \smallsetminus E_t)^{\infty}_{-}$ which we will denote by the same symbol. Consider a J_t–holomorphic curve in $(M \smallsetminus E_t)^{\infty}_{-}$ which has genus zero, degree d, and s^- negative ends asymptotic to multiples of γ_1 such that the i^{th} such end covers γ_1 a total of b_i times. The virtual dimension of the moduli space represented by this curve is

$$(n-3)(2-s^-) + 6d - \sum_{i=1}^{s^-} \left(2b_i + (n-1)\left(2\left\lfloor \frac{r}{S^2} \right\rfloor + 1\right)\right). \tag{13}$$

Now, let J_t be a smooth family of almost-complex structures such that J_t belongs to $\mathcal{J}_t$ for all $t \in [1/S, 1]$. We define $\mathcal{K}_t$ to be the moduli space of somewhere injective J_t–holomorphic planes in $(M \smallsetminus E_t)^{\infty}_{-}$ which have finite energy, degree d, and whose negative end is asymptotic to $\gamma_1^{(3d-1)}$. Since $S > \sqrt{3d-1}$, the index formula above implies that each $\mathcal{K}_t$ has virtual dimension

$$(n-3) + 6d - (2(3d-1) + (n-1)) = 0.$$

3.1 The space $\mathcal{K}_{1/S}$.

Here we prove the following result.

Proposition 3.1. *For sufficiently large $S > 0$ and every regular almost-complex structure $J_{1/S}$ in $\mathcal{J}_{1/S}$, the corresponding moduli space $\mathcal{K}_{1/S}$ is an oriented, compact, zero-dimensional manifold whose oriented cobordism class is nontrivial.*

It suffices to prove that the conditions on $\mathcal{K}_{1/S}$ are satisfied for any regular almost-complex structure $J_{1/S}$ in $\mathcal{J}_{1/S}$. We will restrict our attention to the subset $\bar{\mathcal{J}}_{1/S}$ of $\mathcal{J}_{1/S}$ consisting of almost-complex structures which are invariant under the $\mathbb{T}^{n-2}$–action on M. Note that for $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$ the submanifold $(\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\} \subset M$ is $J_{1/S}$-holomorphic. Hence, Theorem 2.16 implies that, for a suitable choice of the restriction of $J_{1/S}$ to $(\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\}$, the space $\mathcal{K}_{1/S}$ is nonempty. In particular, the theorem yields a curve with image in

$$((\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\})^{\infty}_{-} \subset (M \smallsetminus E_{1/S})^{\infty}_{-}$$

that represents a class in $\mathcal{K}_{1/S}$.

Proposition 3.2. *For $J_{1/S}$ in $\bar{\mathcal{J}}_{1/S}$, let F be an immersed $J_{1/S}$-holomorphic curve in $(M \smallsetminus E_{1/S})^{\infty}_{-}$ whose image is contained in the subset $((\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\})^{\infty}_{-}$. The linearized normal bundle ν for the image of F splits as a sum of holomorphic line bundles. In particular, $\nu = H \oplus V_3 \oplus \ldots \oplus V_n$, where H is the subbundle of vectors parallel to $\{z_j = 0 \mid j = 3, \ldots n\}$ and V_j is the subbundle of vectors along F parallel to the z_j factor.*

Proof. Let π denote the natural projection $M \to \mathbb{C}P^2$. This is of course holomorphic with respect to the standard integrable complex structure i. Let $J_{1/S}$ be our (stretched) almost-complex structure on $(M \smallsetminus E_{1/S})^{\infty}_{-}$ and also the induced complex structure on ν.

It is clear that H is a holomorphic subbundle. Since the derivative of the projection is holomorphic along F the fibers of the V_j are complex. But we can choose a connection on V_j whose horizontal subspaces are all invariant under rotations in the z_j plane. Then if v is the lift of a vector from the base F, the projection of $J_{1/S}v$ to the fibers is also invariant under rotation. But the component of this projection in the $H \oplus_{i \neq j} V_i$ factors varies linearly with $z_j \in V_j$ and is fixed under rotation, and so must be identically zero. It follows that V_j is a holomorphic subbundle as required.

□

Now, for $J_{1/S}$ in $\bar{\mathcal{J}}_{1/S}$ which restrict to generic almost-complex structures on $((\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\})^{\infty}_{-}$, curves which represent classes in $\mathcal{K}_{1/S}$ and whose images lie in $((\mathbb{C}P^2 \smallsetminus E(1/S,1)) \times \{0\})^{\infty}_{-}$ are immersed, and by Proposition 3.2 the linearized operator along such curves splits. Thus we can apply the (four-dimensional) automatic transversality results of [21] to

conclude that such curves are regular (in particular our curves satisfy the hypotheses of [21] Theorem 1 since the c_N term there is negative).

We recall that to compute the difference in orientation of two curves in the 0 dimensional moduli space $\mathcal{K}_{1/S}$ we must identify their normal bundles and examine a family of linear almost-complex structures interpolating between the two induced structures. The difference is given by a sum of crossing numbers evaluated at points where the associated Cauchy-Riemann operators have nontrivial cokernel (see for example [16], Remark 3.2.5). But the automatic regularity here means that there are no such irregular almost-complex structures and thus all curves have the same orientation.

Observe that if $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$ is regular for all curves representing classes in $\mathcal{K}_{1/S}$, then the images of these curves are *all* contained in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$. This is because any such curve whose image is not contained in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ would have deformation index zero but, by invariance, would appear in a family of curves of dimension at least one. Hence, it follows from Theorem 2.16 and Proposition 3.2 that to prove Proposition 3.1 it suffices for us to establish the existence of a $J_{1/S}$ in $\bar{\mathcal{J}}_{1/S}$ which is regular for $\mathcal{K}_{1/S}$.

A curve which represents a class in $\mathcal{K}_{1/S}$ will be called *orbitally simple* if it intersects at least one orbit of the $\mathbb{T}^{n-2}$–action exactly once. There is an open and dense subset of $\bar{\mathcal{J}}_{1/S}$ for which all orbitally simple curves for the corresponding spaces $\mathcal{K}_{1/S}$ are regular. This follows from the standard methods, exactly as in, say, [16] §3.2, Here, the condition of orbital simplicity replaces the assumption in [16] that all curves are simple. In particular, the analogue of Proposition 3.2.1 of [16] allows one to construct sections of bundles over $(M \setminus E_{1/S})^\infty_-$ which are both $\mathbb{T}^{n-2}$–invariant and, when restricted to the image of a curve, have support contained in the neighborhood of a single point.

Thus, to detect the desired almost-complex structure $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$ it will suffice to find an open subset of $\bar{\mathcal{J}}_{1/S}$ such that every curve for the corresponding spaces $\mathcal{K}_{1/S}$ is either contained in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ or is orbitally simple. In fact, as orbital simplicity is an open property (by compactness), it will suffice to construct a single such almost-complex structure.

Proposition 3.3. *For a suitable $J_{1/S} \in \bar{\mathcal{J}}_{1/S}$, all curves which represent classes in $\mathcal{K}_{1/S}$ and which do not lie entirely in $((\mathbb{C}P^2 \setminus E(1/S,1)) \times \{0\})^\infty_-$ must intersect some orbit of the $\mathbb{T}^{n-2}$–action exactly once.*

Proof. For a real number a slightly larger than 1, let Σ_a denote the hyper-

surface $\{S|z_1|^2 + |z_2|^2 = a^2\} \subset (M \setminus E_{1/S})^\infty_-$ which divides $(M \setminus E_{1/S})^\infty_-$ into two regions; V which contains the cylindrical concave end corresponding to $\partial E_{1/S}$, and $W = \{S|z_1|^2 + |z_2|^2 > a^2\}$. We will work with $\mathbb{T}^{n-2}$–invariant almost-complex structures on $(M \setminus E_{1/S})^\infty_-$ for which the projection onto $\{z_j = 0 \mid j = 3, \dots, n\}$ is holomorphic on W. Let J_N denote a sequence of such almost-complex structures which are stretched to a length N along Σ_a. We observe that Σ_a contains a $2(n-2)$ parameter family of Reeb orbits corresponding to translations of $a\gamma_1$.

Arguing by contradiction, suppose that for all such almost-complex structures J_N there exist curves f_N which represent classes in $\mathcal{K}_{1/S}$ and intersect $\mathbb{T}^{n-2}$–orbits in multiple points, or not at all. Then on W the curves f_N project with a degree $r_N > 1$ onto curves h_N lying in $\{z_j = 0 \mid j = 3, \dots, n\}$. The sequence of r_N is bounded by d, as the f_N intersect the line at infinity at least r_N times. So, taking a subsequence, we may assume that $r = r_N$ is independent of N.

Now, taking a limit as $N \to \infty$ our curves f_N will converge to a holomorphic building whose component in W has negative ends asymptotic to translations of $a\gamma_1$. Since the component in V has a negative end asymptotic to $\gamma_1^{(3d-1)}$ we may assume, since our curves have positive area, that for a sufficiently close to 1 the negative ends in W cover the orbits a total of at least $3d - 1$ times.

The projected curves h_N have bounded area and so will also converge to a holomorphic building which includes a curve h with image in

$$(\{z_j = 0 \mid j = 3, \dots, n\} \cap \{S|z_1|^2 + |z_2|^2 > a^2\})^\infty_-.$$

Working with $\mathbb{T}^{n-2}$–invariant structures we may still assume that the limiting almost-complex structure on $(\{z_j = 0 \mid j = 3, \dots, n\} \cap \{S|z_1|^2 + |z_2|^2 > a^2\})^\infty_-$ is regular (indeed, any almost-complex structure preserving $\{z_j = 0 \mid j = 3, \dots, n\}$ can be extended to W as a $\mathbb{T}^{n-2}$–invariant one).

Suppose that h has degree e and s^- negative ends with the i^{th} such end covering $a\gamma_1$ a total of c_i times. Using formula (4), the deformation index of h is

$$\text{index}(h) \quad = \quad -2 + 6e - 2\sum_{i=1}^{s^-} c_i$$

which is nonnegative only if $\sum_{i=1}^{s^-} c_i \leq 3e - 1$. Now, the degree e of h is equal to $\frac{d}{r}$, so the negative ends of the limits of our curves f_N cover the translates

of $a\gamma_1$ a total of $r\sum_{i=1}^{s^-} c_i \leq r(3e-1) = 3d - r$ times. Thus, if $r > 1$ then we have a contradiction as required.

□

3.2 A compact cobordism

For a generic choice of the family J_t, the set $\mathcal{K} = \{\mathcal{K}_t \mid t \in [1/S, 1]\}$ is an oriented 1-dimensional manifold with boundary. By [2], $\mathcal{K}$ is compact modulo convergence to equivalence classes of holomorphic buildings in the $(M \smallsetminus E_t)^\infty_-$. A priori, such buildings could include multiply covered curves, that is, curves not included in the definition of $\mathcal{K}$. In this section we prove

Proposition 3.4. *The family $\mathcal{K}$ is compact.*

Proof. Let $\mathbf{F}$ be a holomorphic building in $(M \smallsetminus E_t)^\infty_-$ representing a limit point of $\mathcal{K}$. Then $\mathbf{F}$ consists of a finite collection of holomorphic curves with connected domains and images in either $(M \smallsetminus E_t)^\infty_-$ or SE_t. Each of these curves has virtual index zero and their compactifications fit together to form a continuous map $\overline{\mathbf{F}}$ from the unit disc to $M \smallsetminus E_t$ which takes the boundary circle to $\gamma_1^{(3d-1)}$. Arguing as in Section 2.4, one can also show that the curves of $\mathbf{F}$ with image in $(M \smallsetminus E_t)^\infty_-$ have negative ends which are asymptotic only to multiples of γ_1, and the curves with image in SE_t are multiply covered holomorphic cylinders over γ_1. Moreover, as inherited by the curves in $\mathcal{K}$, there is a unique curve of $\mathbf{F}$ at the bottom level, and it has a single negative end which covers γ_1 precisely $3d-1$ times. This is because any curves without negative ends necessarily have positive deformation index.

Our first claim is that there is a single curve of $\mathbf{F}$ with image in $(M \smallsetminus E_t)^\infty_-$, and that it is somewhere injective. Arguing by contradiction, suppose that there are $K > 1$ such curves. Then they have degrees $d_1, \dots, d_K$ with $d_1 + \dots + d_K = d$ and we may assume that the ends of the i^{th} component cover γ_1 a total number t_i times. Here we allow for some $t_i = 0$, which corresponds to rational curves without (nonremovable) punctures which bubble off in $(M \smallsetminus E_t)^\infty_-$. Since the curves of $\mathbf{F}$ with image in SE_t have at least as many positive ends as negative ends, when counted with multiplicity, and the curve of lowest level has a single negative end which covers γ_1 a total of $3d-1$ times, we have $t_1 + \cdots + t_K \geq 3d-1$. The assumption that $K > 1$ then implies that there must exist a $j \in [1, K]$ for which $t_j \geq 3d_j$. Suppose the corresponding curve F of $\mathbf{F}$ is an r-fold cover of a somewhere injective curve $\widetilde{F}$ (if F is not

a multiple cover then we allow $r = 1$ and $\widetilde{F} = F$ here). Then $\widetilde{F}$ has degree $\tilde{d} = \frac{d_j}{r}$ and has some number $\tilde{s}^-$ of negative ends, the i^{th} of which covers γ_1 a total of $\tilde{c}_i$ times. Hence, $\sum_{i=1}^{\tilde{s}^-} \tilde{c}_i = \frac{t_j}{r}$ and the index formula (13) for $\widetilde{F}$ becomes

$$\begin{aligned} \mathrm{index}(\widetilde{F}) &= (n-3)(2-\tilde{s}^-) + 6\tilde{d} - \sum_{i=1}^{\tilde{s}^-}(2\tilde{c}_i + (n-1)) \\ &= (2n-4)(1-\tilde{s}^-) - 2 + \frac{2}{r}(3d_j - t_j) \\ &\leq -2. \end{aligned}$$

For generic choices of almost-complex structure we then have a contradiction.

Thus, there is a single curve G of $\mathbf{F}$ with image in $(M \smallsetminus E_t)^\infty_-$. Suppose that this curve has a total of s^- negative ends, each covering γ_1 a number c_i times. If G is an r-fold cover of a somewhere injective curve $\widetilde{G}$ with $\tilde{s}^-$ negative ends then, as above, the index formula (13) gives

$$\mathrm{index}(\widetilde{G}) = (2n-4)(1-\tilde{s}^-) - 2 + \frac{2}{r}(3d - \sum_{i=1}^{s^-} c_i).$$

But as $\sum_{i=1}^{s^-} c_i \geq 3d - 1$, we notice that the index can exceed -2 only if $\tilde{s}^- = 1$ and $r = 1$ and $\sum_{i=1}^{s^-} c_i = 3d - 1$. In other words, the unique curve G of $\mathbf{F}$ with image in $(M \smallsetminus E_t)^\infty_-$ is somewhere injective and does indeed lie in $\mathcal{K}$.

Now assume that $\mathcal{K}$ is not compact and that $\mathbf{F}$ represents a limit point not contained in $\mathcal{K}$. Then $\mathbf{F}$ is a holomorphic building of height $k > 1$. As described above, the unique curve of $\mathbf{F}$ of level one has a single negative end which covers γ_1 precisely $3d - 1$ times. Since $k > 1$, this curve, say H, has image in SE_t. The compactifications of G, H, and the other curves of $\mathbf{F}$ with image in SE_t fit together to form a curve of genus zero. Hence, the curves with level in $[1, k)$ must each have one positive and one negative end. If follows that the curves of $\mathbf{F}$ with image in SE_t are all cylindrical, and so $\mathbf{F}$ and G represents the same element of $\mathcal{K}$. This contradiction affirms that $\mathcal{K}$ is compact.

□

3.3 The completion of the proof

Theorem 1.2 now follows almost immediately from Propositions 3.1 and 3.4. For any $S > 0$ with $S^2 \in \mathbb{R} \smallsetminus \mathbb{Q}$, and any positive integer d satisfying $d(3d-1) < S^2$, it follows from Proposition 3.4 that for a generic family J_t the space $\mathcal{K} = \{\mathcal{K}_t \mid t \in [1/S, 1]\}$ is a compact, oriented, 1-dimensional manifold whose boundary is $\mathcal{K}_{1/S} \cup \mathcal{K}_1$. Proposition 3.1 then implies that the moduli space $\mathcal{K}_1$ is nonempty. Hence, there exists a holomorphic plane in $(M \smallsetminus E_1)^{\infty}_{-}$ whose negative end is asymptotic to $\gamma_1^{(3d-1)}$. The symplectic area of this curve is positive and equal to $d\pi R^2 - (3d-1)\pi$. Thus, $R^2 > \frac{3d-1}{d}$, and taking the limit as d (and hence S) goes to ∞ we have $R^2 \geq 3$.

4 Symplectic embeddings

In this section we prove Theorems 1.3 and 1.6.

4.1 The proof of Theorem 1.6

Let $\Sigma(\epsilon)$ be a punctured torus, i.e., a surface of genus one with one boundary component, equipped with a symplectic form of total area ϵ. The Main Lemma of [7] implies the following.

Proposition 4.1. *For any $S, \epsilon > 0$, there exists a symplectic embedding of $B^{2(n-1)}(S)$ into $\Sigma(\epsilon) \times \mathbb{R}^{2(n-2)}$.*

To establish Theorem 1.6 it suffices to find an embedding of $\Sigma(\epsilon) \times B^2(1)$ into $B^2(\sqrt{R}) \times B^2(\sqrt{R})$, where $R > 2$ is fixed and ϵ can be arbitrarily small. The existence of such an embedding is essentially contained in Lemma 3.1 of [7]. We review the construction here for the sake of completeness and because it will play an important role in the proof of Theorem 1.3.

Fix $R = 2 + 2\delta$ for $\delta > 0$. We begin with the following elementary result.

Lemma 4.2. *For every $\delta > 0$ there is a nonnegative function H whose support is contained in $B^2(\sqrt{2+2\delta})$, whose maximum value is less than $\pi + \delta$, and whose time-t Hamiltonian flow, ϕ^t_H, satisfies*

$$\phi^t_H(B^2(1)) \subset B^2(\sqrt{(1+t)(1+\delta/\pi)})$$

and

$$\phi^1_H(B^2(1)) \cap B^2(1) = \emptyset.$$

Proof. Let U be any set whose closure is contained in the interior of the square $[0, \sqrt{\pi+\delta}] \times [0, \sqrt{\pi+\delta}] \subset \mathbb{R}^2$. The time-$t$ Hamiltonian flow of the function $K(x,y) = (\sqrt{\pi+\delta})x$ on $\mathbb{R}^2$ is given by

$$\phi_K^t(x,y) = (x, y + t\sqrt{\pi+\delta}).$$

(We use the convention that the Hamiltonian vectorfield, X_K, of K is defined by the equation $i_{X_K}\omega_0 = -dK$.) Hence, for all $t > 0$ the set $\phi_K^t(U)$ is contained in the interior of the rectangle $[0, \sqrt{\pi+\delta}] \times [0, (1+t)\sqrt{\pi+\delta}]$ and $\phi_K^1(U) \cap U = \emptyset$. Cutting K off appropriately near $\bigcup_{t\in[0,1]} \phi_K^t(U)$, we get a nonnegative function $\hat{K}$ whose Hamiltonian flow still has these properties, but is now supported in $[0, \sqrt{\pi+\delta}] \times [0, 2\sqrt{\pi+\delta}]$ and satisfies $\max(\hat{K}) < \pi + \delta$.

One can construct a symplectic diffeomorphism ψ of $\mathbb{R}^2$ which maps $[0, \sqrt{\pi+\delta}] \times [0, 2\sqrt{\pi+\delta}]$ into $B^2(\sqrt{2+2\delta})$ and for $t \in [0,1]$ maps arbitrarily large subsets of each rectangle $[0, \sqrt{\pi+\delta}] \times [0, (1+t)\sqrt{\pi+\delta}]$ onto balls centered at the origin. (Such maps are described and illustrated explicitly in Section 3.1 of [18].) We choose these arbitrarily large subsets of the rectangles $[0, \sqrt{\pi+\delta}] \times [0, (1+t)\sqrt{\pi+\delta}]$ so that they contain $\phi_K^t(U)$ for all $t \in [0,1]$. Setting $U = \psi^{-1}(B^2(1))$ and $H = \hat{K} \circ \psi$, we are done. ☐

Remark 4.3. It is clear from the definition of H in terms of $\hat{K}$ that for all $t \in [0,1]$ the distance between $\phi_H^t(B^2(1))$ and the boundary of $B^2(\sqrt{(1+t)(1+\delta/\pi)})$ is greater than zero and of order δ.

Consider an immersion i_δ of $\Sigma(\epsilon)$ into $\mathbb{R}^2$, as sketched in Figure 1, with the following properties:

- the double points are concentrated in an arbitrarily small region around the origin $(0,0)$.
- the vertical and horizontal sections crossing at $(0,0)$ are arbitrarily thin (and hence arbitrarily long if need be).
- the areas of the regions A and B are both equal to $\pi + 2\delta$.

By the last of these properties we may assume that, for sufficiently small $\epsilon > 0$, the immersion lies in a region symplectomorphic to $B^2(\sqrt{2+2\delta})$.

Let I_δ^0 be the symplectic immersion of $\Sigma(\epsilon) \times B^2(1)$ into $B^2(\sqrt{2+2\delta}) \times B^2(\sqrt{2+2\delta})$ which acts as i_δ on the first factor and as inclusion on the

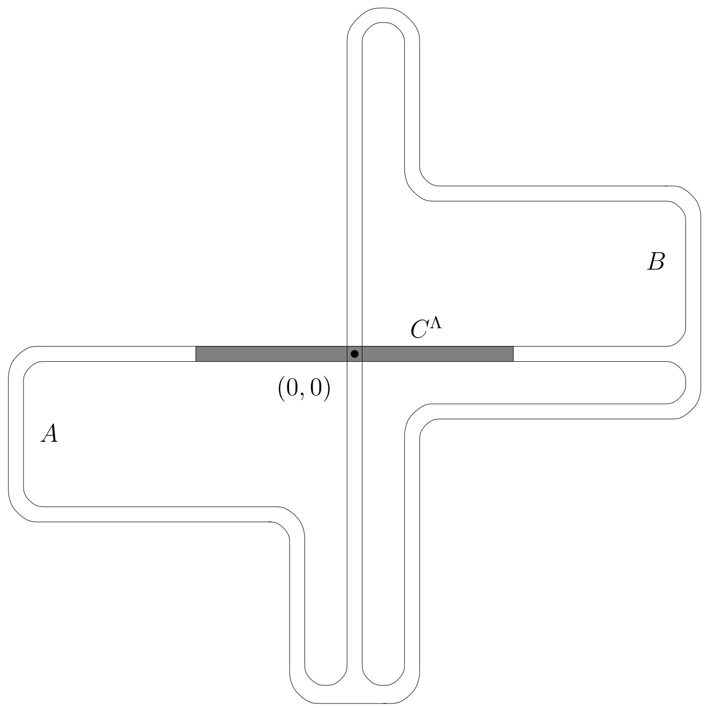

Figure 1: The symplectic immersion i_δ of the punctured torus.

second. We now alter the image of I^0_δ to obtain the desired embedding. In what follows, we will use coordinates (x_1, y_1) on the plane containing the first copy of $B^2(\sqrt{2+2\delta})$ and coordinates (x_2, y_2) on the plane containing the second copy. The projection from $\mathbb{R}^4$ to the x_1y_1-plane will be denoted by pr_1.

The self-intersections of I^0_δ project under pr_1 to the self intersections of i_δ. The x_1-coordinates of these points take values in an interval of the form $[-\lambda, \lambda]$. For $\Lambda > \lambda$, let C^Λ be the horizontal portion of the image of i_δ which passes through the origin and whose first coordinates satisfy $x_1 \in [-\Lambda, \Lambda]$. To remove the intersections of I^0_δ, we consider the Hamiltonian $\hat{H} = \chi(x_1)H(x_2, y_2)$ where H is the Hamiltonian from Lemma 4.2 and χ is a bump-function which equals 1 for $|x_1| \leq \lambda$ and equals 0 for $|x_1| \geq \Lambda$. The time-1 Hamiltonian flow of $\hat{H}$ is

$$\phi_{\hat{H}}(x_1, y_1, x_2, y_2) = (x_1, y_1 + \chi'(x_1)H(x_2, y_2), \phi_H^{\chi(x_1)}(x_2, y_2)).$$

Let I^1_δ be the symplectic immersion of $\Sigma(\epsilon) \times B^2(1)$ into $B^2(\sqrt{2+2\delta}) \times$

$B^2(\sqrt{2+2\delta})$ obtained by applying $\phi_{\hat{H}}$ to $C^\Lambda \times B^2(1)$. Clearly, I^1_δ agrees with I^0_δ away from $i_\delta^{-1}(C^\Lambda) \times B^2(1)$, and shares none of the original double points of I^0_δ. The immersion I^1_δ can only have new double points in $\phi_{\hat{H}}(C^\Lambda_\pm \times B^2(1))$ where C^Λ_+ and C^Λ_- are the portions of C^Λ corresponding to points with x_1-values in $[\lambda, \Lambda]$ and $[-\Lambda, -\lambda]$, respectively. We now show that, for an appropriate choice of i_δ and χ, I^1_δ can be adjusted on $\phi_{\hat{H}}(C^\Lambda_\pm \times B^2(1))$ so that no new double points occur.

In the x_1y_1-plane, $\phi_{\hat{H}}$ only moves points in $C^\Lambda_\pm$ and does so only in the y_1–direction. Moreover, the maximum displacement in this direction is bounded from above by

$$\left| \int_0^1 \chi'(x_1) \max(H)\, dx_1 \right| < \max(|\chi'|)(\pi + \delta).$$

Hence, the image of $\phi_{\hat{H}}(C^\Lambda_+ \times B^2(1))$ under pr_1 is contained in the set

$$C^\Lambda_+ + ([\lambda, \Lambda] \times [0, \max(|\chi'|)(\pi + \delta)])\,,$$

and the projection of $\phi_{\hat{H}}(C^\Lambda_- \times B^2(1))$ is contained in

$$C^\Lambda_- + ([-\Lambda, -\lambda] \times [-\max(|\chi'|)(\pi + \delta), 0])\,.$$

Choosing the width of C^Λ to be sufficiently small, and $\max(|\chi'|)$ sufficiently close to $\frac{1}{\Lambda - \lambda}$, we may assume that both $\phi_{\hat{H}}(C^\Lambda_\pm \times B^2(1))$ project to regions in the x_1y_1-plane whose area is less than $\pi + 2\delta$ and hence less than the area of each of the regions A and B. Acting on $\phi_{\hat{H}}(C^\Lambda_- \times B^2(1))$ and $\phi_{\hat{H}}(C^\Lambda_+ \times B^2(1))$ by another symplectic diffeomorphism which acts nontrivially only in the x_1y_1–directions, we may then ensure that they are mapped by pr_1 into A and B, respectively. The resulting symplectic immersion therefore has no double points and is the desired embedding of Theorem 1.6.

4.2 The proof of Theorem 1.3

Scaling things appropriately, the argument above yields a symplectic embedding of $\Sigma(\epsilon) \times B^2(r)$ into $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ for any $r < 1$ provided that ϵ is sufficiently small. In this section, we show that the previous embedding procedure can be refined to obtain a symplectic embedding of $\Sigma(\epsilon) \times B^2(r)$ into $B^4(\sqrt{3})$. This will prove Theorem 1.3 which implies that Theorem 1.1 is sharp.

Remark 4.4. The bi-disc $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ can be symplectically embedded into $B^4(2)$, by inclusion. The second Ekeland-Hofer capacity implies that this is optimal in the sense that $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ can not be symplectically embedded into a smaller ball.

As in the proof of Theorem 1.6, we start with a symplectic immersion I^0 of $\Sigma(\epsilon) \times B^2(r)$ into $B^2(\sqrt{2}) \times B^2(\sqrt{2})$ which acts by an immersion $i \colon \Sigma(\epsilon) \hookrightarrow B^2(\sqrt{2})$ in the first factor, and by inclusion on the second factor. The immersion i is chosen so that for some $\lambda > 0$ we have:

- the vertical and horizontal crossing portions of the image have length equal to $2\pi/\lambda$.
- the regions A and B have areas in the interval $(\pi r^2, \pi)$ and all but an arbitrarily small amount of this area is concentrated within a distance λ of the horizontal crossing portion, C.

See Figure 2.

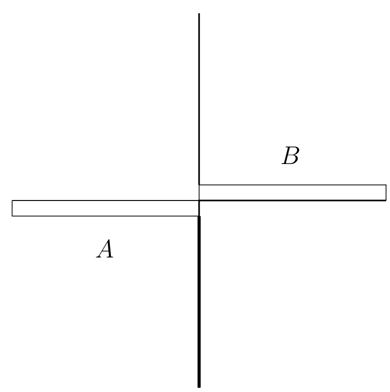

Figure 2: The symplectic immersion i, from a distance.

Set $\Lambda = \pi/\lambda$, so that $C^\Lambda = C$. Defining χ and $\hat{H}$ as in the proof of Theorem 1.6, we remove the double points of I^0 by applying the time-1 flow of $\hat{H}$ to $C \times B^2(r)$ to obtain a new immersion I^1. Choosing λ and $\left(\max(|\chi'|) - \frac{1}{\Lambda - \lambda}\right)$ to be sufficiently small, we may assume that the projection pr_1 maps $\phi_{\hat{H}}(C \times B^2(r))$ to the interior of the following region of the x_1y_1-plane

$$\mathbf{C} = C + \{([-\pi/\lambda, -\lambda] \times [-\lambda, 0]) \cup ([\lambda, \pi/\lambda] \times [0, \lambda])\}.$$

When the width of C is small enough, the area of $\mathbf{C}$ is less than 2π. As in the proof of Theorem 1.6 we can then apply a suitable symplectic map to $\phi_{\hat{H}}(C \times B^2(r))$ to shift the relevant parts of its projection into A and B and hence obtain a symplectic embedding of $\Sigma(\epsilon) \times B^2(r)$ into $B^2(\sqrt{2}) \times B^2(\sqrt{2})$.

We now refine this embedding procedure by choosing a new immersion of $\Sigma(\epsilon)$. We begin by analyzing the fibers of the projection map pr_1 acting on $I^1(\Sigma(\epsilon) \times B^2(r))$. The points of $I^1(\Sigma(\epsilon) \times B^2(r))$ not in $\phi_{\hat{H}}(C \times B^2(r))$ belong to fibres which can all be identified with $B^2(r)$. The points in $\phi_{\hat{H}}(C \times B^2(r))$ belong to fibres of pr_1 determined by the x_1-component of their projections. That is, the fibres of pr_1 corresponding to a fixed value of x_1 can all be identified with a fixed subset of $B^2(\sqrt{2})$, which we denote by $F(x_1)$. For $|x_1| \leq \lambda$, $F(x_1)$ is a fixed subset of the interior of $B^2(\sqrt{2})$. For $\lambda < |x_1| \leq \pi/\lambda$, each $F(x_1)$ is contained in the interior of $B^2(\sqrt{1+\chi(x_1)})$ (see Lemma 4.2). We can choose the bump function $\chi(x_1)$ so that on $[-\pi/\lambda, -\lambda] \cup [\lambda, \pi/\lambda]$ it is arbitrarily C^0-close to the function $x_1 \mapsto 1 - \frac{|x_1|-\lambda}{\pi/\lambda-\lambda}$. For all sufficiently small $\lambda > 0$ we may then assume that $F(x_1)$ is contained in the interior of $B^2(\sqrt{2-|x_1|\lambda/\pi})$ for all $x_1 \in [-\pi/\lambda, \pi/\lambda]$. By Remark 4.3, the distance from $F(x_1)$ to the boundary of $B^2(\sqrt{1+\chi(x_1)})$ is bounded from below by a positive constant which is independent of λ and which goes to zero as r approaches $\sqrt{2}$. For χ as above, and λ sufficiently small, we may assume the same is true of the distance from $F(x_1)$ to the boundary of $B^2(\sqrt{2-|x_1|\lambda/\pi})$. In this case we denote the lower bound for this distance by D_r.

We now apply a symplectomorphism w to the x_1y_1-plane which winds $\mathbf{C}$ around itself as sketched in Figure 3. This winding is nearly tight but includes small gaps (represented by the thicker lines in Figure 3) to accommodate the winding of the rest of $i(\Sigma(\epsilon))$. Since the area of $\mathbf{C}$ is less than 2π, for sufficiently small $\epsilon > 0$ we may assume that the image of $i(\Sigma(\epsilon)) \cup \mathbf{C}$ under the winding map still lies in the ball $B^2(\sqrt{2})$. Replacing the embedding i in the previous construction with the composition $w \circ i$, we get a new symplectic embedding

$$I_w \colon \Sigma(\epsilon) \times B^2(r) \hookrightarrow B^2(\sqrt{2}) \times B^2(\sqrt{2}).$$

We now show that the image of I_w is contained in $B^4(\sqrt{3})$.

Let $(z_1, z_2) \in \mathbb{C}^4$ be any point in the image of I_w. Then $z_1 = w(x_1, y_1)$ for a unique point (x_1, y_1) in $i(\Sigma(\epsilon))$ and z_2 belongs to $F(x_1)$. Since $w(i(\Sigma(\epsilon)) \cup \mathbf{C}) \subset B^2(\sqrt{2})$ we have

$$|z_1| < \sqrt{2}. \tag{14}$$

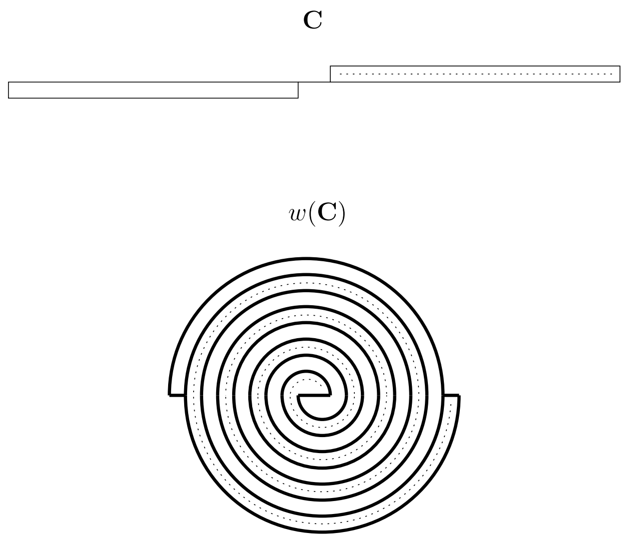

Figure 3: The winding map w acting on $\mathbf{C}$.

By the analysis of the fibres $F(x_1)$ above, we have

$$|z_2| \leq \sqrt{2 - |x_1|\lambda/\pi} - D_r. \tag{15}$$

On the other hand it follows from the definition of the winding map w that

$$|z_1| = \sqrt{\frac{2\lambda}{\pi}|x_1|} + O(\lambda) + O(\epsilon\lambda). \tag{16}$$

The first approximation here comes from equating the area of the portion of C determined by x_1, $2|x_1|\lambda$, with $\pi|z_1|^2$. The error terms of (16) correspond, respectively, to the discrepancy caused by the width of $\mathbf{C}$, and the discrepancy caused by the gap in the winding.

Together, equations (15) and (16) yield

$$(|z_2| + D_r)^2 + \frac{1}{2}(|z_1| - O(\lambda) - O(\epsilon\lambda))^2 \leq 2 \tag{17}$$

The fact that D_r is positive and independent of λ, for sufficiently small $\lambda > 0$, implies that

$$|z_2|^2 + \frac{1}{2}|z_1|^2 \leq 2. \tag{18}$$

Together, inequalities (14) and (18) then yield

$$|z_1|^2 + |z_2|^2 \leq 3, \tag{19}$$

as desired.

References

[1] F. Bourgeois, A Morse-Bott approach to contact homology. Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), 55–77, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, 2003.

[2] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, Compactness results in symplectic field theory, *Geom. Topol.*, **7** (2003), 799–888.

[3] D. Dragnev, Fredholm theory and transversality for noncompact pseudoholomorphic maps in symplectizations, *Comm. Pure Appl. Math.*, **57** (2004), 726-763.

[4] I. Ekeland and H. Hofer, Symplectic topology and Hamiltonian dynamics II, *Math. Z.*, **203** (1990), 553-567.

[5] Y. Eliashberg, A. Givental and H. Hofer, Introduction to symplectic field theory, GAFA 2000 (Tel Aviv, 1999), *Geom. Funct. Anal.*, 2000, Special Volume, Part II, 560–673.

[6] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, *Inv. Math.*, **82** (1985), 307–347.

[7] L. Guth, Symplectic embeddings of polydisks, *Inv. Math*, **172** (2008), 477–489.

[8] J. Harris and I. Morrison, Moduli of Curves, Springer 1998.

[9] R. Hind, Hamiltonian displacement of bidisks inside cylinders, preprint.

[10] H. Hofer, K. Wysocki and E. Zehnder, A characterisation of the tight three-sphere, *Duke Math. J.*, **81** (1995), 159–226.

[11] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations I: Asymptotics, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, **13** (1996) , 337–379.

[12] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations II: Embedding controls and algebraic invariants, *Geom. Funct. Anal.*, **5** (1995), 337–379.

[13] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations III: Fredholm theory, *Topics in nonlinear analysis*, 381-475, Prog. Nonlinear Differential Equations Appl., 35, Birkhäuser, Basel, 1999.

[14] M. Hutchings, An index inequality for embedded pseudoholomorphic curves in symplectizations, *J. Eur. Math. Soc. (JEMS)*, 4 (2002), no. 4, 313–361.

[15] M. Hutchings, The embedded contact homology index revisited, *New perspectives and challenges in symplectic field theory*, 263–297, CRM Proc. Lecture Notes, 49, Amer. Math. Soc., Providence, RI, 2009.

[16] D. McDuff and D. Salamon, J-holomorphic curves and symplectic topology. American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.

[17] J. Robbin and D. Salamon, The Maslov index for paths, *Topology*, **32** (1993), 827–844.

[18] F. Schlenk, Embedding problems in symplectic geometry De Gruyter Expositions in Mathematics 40. Walter de Gruyter Verlag, Berlin. 2005.

[19] V. Shevchishin, Pseudoholomorphic curves and the symplectic isotopy problem, preprint math.SG/0010262.

[20] L. Traynor, Symplectic packing constructions, *J. Diff. Geom.*, **42** (1995), 411–429.

[21] C. Wendl, Automatic transversality and orbifolds of punctured holomorphic curves in dimension four, to appear in *Comment. Math. Helv.*, preprint arXiv:0802.3842.